Regular Abelian groups and elements

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Abstract

Starting with Zelmanowitz (1972) definition of regular elements in modules and the special case of unit-regular elements (Chen, Nicholson, Zhou, 2018), the (unit-) regular elements (and modules) are determined for $\mathbb{Z}$-modules, that is, for Abelian groups.

1 Introduction

Abelian group theory was (and sometimes still is) "one of the principal motives of new research in module theory (e.g. for every particular theorem on Abelian groups one can ask over what rings the same result holds) - L. Fuchs".

However, sometimes the reversed direction may not be trivial: notions which were defined for modules (not as generalizations of Abelian groups) are sometimes hard to determine for modules over the ring of integers, i.e. for Abelian groups.

"Discovered (around 1935) by John von Neumann in connection with his work on continuos geometry and operator algebras, the (von Neumann) regular rings were defined by the following condition: for any $a \in R$, there exists $x \in R$ such that $a = axa$. Equivalently, every principal left ideal is generated by an idempotent (or, every left ideal is a direct summand of ${}_RR$) - T. Y. Lam".

Abelian groups with *regular endomorphism ring* were extensively studied in a 1968 paper by L. Fuchs and K. M. Rangaswamy [5].

In the early 70's three definitions were given for modules, extending von Neumann regular definition for rings.

Zelmanowitz [6] calls a *module* ${}_RM$ regular (*z-regular* in the sequel) if for any $m \in M$ there exists $f \in M^* =: \mathrm{Hom}_R(M, R)$ such that $f(m)(m) = m$ (or $mf(m) = m$ if we write values on the left). This definition lends oneself to elements: the *element* $m \in M$ will be called *regular* if there exists $f \in M^*$ such that $f(m)(m) = m$.

Recently (see [1]), starting with this definition and suggested by G. Ehrlich 1970's papers, an element $m \in M$ was called *unit-regular* if there exists an epimorphism $f \in M^*$ such that $f(m)(m) = m$.

Somewhat broader, Fieldhouse [4] calls an object A regular (*f-regular* in the sequel) if every short exact sequence with A in the second place is pure exact.

Finally, somewhat more restricted, for projectives, Ware [7] calls a (projective) module *regular* if every cyclic submodule is a direct summand. In the sequel, an arbitrary module will be called *w-regular* if every cyclic submodule is a direct summand.

The purpose of this (short) note is to determine the corresponding regular Abelian groups (i.e. to study the particular case $R = \mathbf{Z}$) and the regular and unit-regular elements of an Abelian group. Summarizing, regular Abelian groups are rare phenomenon and (unit-) regular elements may be found only in homogeneous torsion-free reduced groups which have type $(0, 0, ...)$.

Therefore, our note could be considered as a short addendum to [6] and [1].

2 The results

Theorem 1 *(i) The f-regular Abelian groups are precisely the elementary groups (i.e. every element has square-free order).*

(ii) The w-regular Abelian groups turn out to coincide with the f-regular ones.

(iii) Nonzero unit-regular elements exist only in homogeneous reduced torsion-free groups of type $(0, 0, ...)$. *If* G *is such a group, a nonzero element* $a \in G$ *is (unit-) regular in* G *iff* $\langle a \rangle$ *is a direct summand of* G *and the characteristics* $\chi(a) = (0, 0, ...)$.

(iv) There are no (nonzero) z-regular Abelian groups.

Proof. (i) A group G is *f-regular* if every short exact sequence

$$0 \longrightarrow A \longrightarrow G \longrightarrow C \longrightarrow 0$$

is pure exact. Equivalently, every subgroup of G is pure. Since the groups all whose subgroups are pure are precisely the elementary groups, the claim follows.

(ii) A group is *w-regular* (an extension of Ware's definition) if every cyclic subgroup is a direct summand.

Since the w-regular property is closed under subgroups, in a w-regular group all cyclic subgroups must be w-regular.

However, the only w-regular cyclic groups are the simple groups, i.e. the prime order (cyclic) groups. Hence the only w-regular groups are the elementary groups.

(iii) Let G, H be Abelian groups. Since $\mathrm{Hom}(G, H) = 0$ for: (a) torsion G, torsion-free H; (b) divisible G, reduced H (and $\mathbb{Z}$ is torsion-free, reduced) with above notation, $G^* = 0$ for any torsion or divisible group G. Hence, *there are no nonzero (unit)-regular elements in torsion groups or in divisible groups.*

Suppose N is a submodule of a module M and $n \in N$. Notice that if n is regular in M then n is also regular in N. This remark reduces the determination of (unit-) regular elements to *reduced torsion-free groups.* Indeed, assume G is not reduced (i.e. $G = D(G) \oplus R$) and suppose $0 \neq d \in D(G)$, the divisible part. The element d cannot be regular in G, otherwise is would be regular in $D(G)$. Similarly for the torsion part $T(G)$.

In general, if $\alpha : A \to B$ is a homomorphism of torsion-free groups, the characteristics $\chi_A(a) \le \chi_B(\alpha(a))$ for every $a \in A$. Therefore $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$ only if G is homogeneous of type $(0, 0, ...)$ (e.g. G may be free, i.e. a direct sum of $\mathbb{Z}$). Therefore, for what follows, we suppose G is a reduced homogeneous torsion-free group of type $(0, 0, ...)$ and determine the (unit-)regular elements.

Suppose $a \neq 0$ is (unit-)regular in a torsion-free reduced group G. By definition, there is homomorphism $f : G \to \mathbb{Z}$ such that $f(a)a = a$ and since the group is torsion-free, $f(a) = 1$. Notice that if there is a homomorphism $f : G \to \mathbb{Z}$ and $f(a) = 1$, then $\langle a \rangle$ is a direct summand of G with complement $\ker(f)$. Moreover, $\chi(a) = (0, 0, ...)$.

Conversely, suppose $a \neq 0$, $\langle a \rangle$ is a direct summand of G and $\chi(a) = (0, 0, ...)$ in a reduced torsion-free group G, homogeneous of type $(0, 0, ...)$. Since $\mathbf{t}_G(a) = (0, 0, ...) = \mathbf{t}_{\mathbb{Z}}(1)$, $a \longmapsto 1$ extends to a nonzero isomorphism $g : \langle a \rangle_* \to \mathbb{Z}$. But $\langle a \rangle = \langle a \rangle_*$ is a direct summand of G and so g extends to homomorphism $f : G \to \mathbb{Z}$ (surjective since g is already surjective) for which we still have $f(a) = 1$, i.e., $f(a)a = a$. Hence a is (unit-) regular in G.

(iv) From the proof of (iii), we infer that *torsion groups and divisible groups are not regular.* It is also immediate that $\mathbf{Z}$ is not regular (since $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z}$, and the ring $\mathbf{Z}$ is not (von Neumann) regular). Further, since *any submodule of a regular module is regular* (see **(1.4)** [6]), we obtain at once that *torsion-free groups are not regular and (genuine) mixed groups* (i.e. $0 \neq T(G) \neq G$) *are not regular.* Therefore there are no (nonzero) z-regular (Abelian) groups. ∎

Example 2 The only nonzero (unit)-regular elements in $\mathbb{Z}$ are ± 1.

Direct proof. Let $0 \neq n \in \mathbb{Z}$. Then n is (unit-)regular in $\mathbb{Z}$ if there is an (epi)morphism $f \in \mathrm{End}(\mathbb{Z})$ with $n = f(n)n$, i.e. $f(n) = 1$.

Clearly $f(n) = 1$ extends additively to $n\mathbb{Z}$ by $f(nk) = k$ for every $k \in \mathbb{Z}$, so what remains is to extend f from $n\mathbb{Z}$ to the whole $\mathbb{Z}$, that is, to extend f to an endomorphism of $\mathbb{Z}$. Since all endomorphisms of $\mathbb{Z}$ are multiplications by integers, suppose there is $\varepsilon \in \mathrm{End}(\mathbb{Z})$ such that $\varepsilon|_{n\mathbb{Z}} = f : n\mathbb{Z} \to \mathbb{Z}$. Then there is an integer m such that $\varepsilon(x) = mx$ for every $x \in \mathbb{Z}$ and so $\varepsilon(n) = f(n) = 1 = mn$. Hence $m, n \in \{\pm 1\}$. Since both give automorphisms, these are regular and also unit-regular (nonzero) elements.

Since a homomorphism $f \in M^*$ is uniquely given by an element $m \in M$, the *z-regular* condition is equivalent to: for any $f \in \mathrm{Hom}_R(R, M)$ there exists $g \in \mathrm{Hom}_R(M, R)$ such that $f = f \circ g \circ f$. This suggests a Category Theory definition of regular objects and indeed this was considered and studied by S. Dăscălescu et altri in [2].

In closing we mention the paper [3], which, just reading its title "Abelian groups and regular modules", apparently could have a connection with our subject. Actually it has not: groups are regarded as w-regular modules (i.e. modules any of whose cyclic submodules is a direct summand) over their rings of endomorphisms. The study of additive groups of rings has already a long history, starting in the 1940's. The paper formulates a more general problem: to *study the additive groups of modules* (starting from some special classes).

A certain information on the additive group of a w-regular module is given, generalizing results in [5].

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