A rank zero nonzero matrix

W. C. Brown *Matrices over commutative rings*

September 11, 2023

From [1].

Definition 4.1. Let $A \in \mathbb{M}_{m\times n}(R)$ with a commutative ring R. For each $t \in \{1, 2, ..., r\}$, $r = \min(m, n)$, $I_t(A)$ denotes the ideal generated by all $t \times t$ minors of A. Then

$$(0) = I_{r+1}(A) \subseteq I_r(A) \subseteq ... \subseteq I_2(A) \subseteq I_1(A) \subseteq R.$$

Accordingly

$$(0) = Ann_R(R) \subseteq Ann_R(I_1(A) \subseteq Ann_r(I_2(A)) \subseteq ... \subseteq Ann_R(I_r(A)) \subseteq Ann_R((0)) = R.$$

Therefore

Definition 4.10 The *rank of* A, denoted $rk(A)$ is the following integer: $rk(A) = \max\{t : Ann_R(I_t(A)) = (0)\}$.

It follows that

4.11 (d) $rk(A) = 0$ iff $Ann_R(I_1(A)) \neq (0)$ [that is, 0 is the maximum integer t above] iff there exists a nonzero $r \in R$ such that $ra_{ij} = 0$ for all i, j.

Example. $A = 2I_2$ over $\mathbb{Z}_4$.

References

[1] W. C. Brown *Matrices over commutative rings*. Marcel Dekker Inc., 1993.