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Negative clean rings

Abstract

A ring is called *negative clean* if the negative (i.e., the additive inverse) of each clean element is also clean. Clean rings are negative clean.

In this paper, we develop the theory of the negative rings, with special emphasis on finding the clean matrices which have (or have not) clean negatives. Many explicit results are proved for 2×2 matrices and some hard to solve quadratic Diophantive equations are displayed.

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1 Introduction

All rings below are associative with identity.

An element in a ring is called *clean* if it a sum of an idempotent and a unit. When we want to emphasize the idempotent, that is, $a = e + u$ with $e^2 = e$ and unit u, we say that a is *e-clean*. If $e \in \{0, 1\}$ (the trivial idempotents), then a is called *trivially* clean. Otherwise, it is *nontrivially* clean. A clean element is *strongly clean* if $eu = ue$, respectively *uniquely clean* if it has only one clean decomposition. We denote by $U(R)$ the set of units of R, by $\mathrm{Id}(R)$ the set of idempotents of R, by $N(R)$ the set of nilpotents of R, by $\mathrm{cn}(R)$ the set of clean elements of R, by $\mathrm{scn}(R)$ the set of strongly clean elements and by $J(R)$ the Jacobson radical of R.

It is well-known that if a is e-clean, then $1 - a$ is $(1 - e)$-clean. However, *the negative (i.e. the additive inverse) of a clean element may not be clean.*

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As examples show, *the negative of a strongly (or uniquely or both strongly and uniquely) clean element may also be not clean.*

Motivated by this, we introduce the following natural

Definition. A ring is called *negative clean* if the negative of each clean element is also clean (equivalently, $-\mathrm{cn}(R) \subseteq \mathrm{cn}(R)$ and so $\mathrm{cn}(R) = -\mathrm{cn}(R)$). Obviously, clean rings are also negative clean. Since $\mathrm{cn}(\mathbb{Z}) = \{-1, 0, 1, 2\}$, $\mathbb{Z}$ is *not* negative clean.

In this paper, we develop the theory of the negative rings, with special emphasis on the determination of the clean matrices which have (or have not) clean negatives. Many explicit results are given for 2×2 integral matrices.

As in many other studies which involve integral 2×2 matrices, some problems reduce to hard to solve Number Theory problems.

While it is easy to show that for any unit $U \in \begin{bmatrix} 2\mathbb{Z}+1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z}+1 \end{bmatrix}$, $-(I_2 + U)$ is not (nontrivially) clean, it is far more difficult to obtain such results for units in $\begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z}+1 \\ 2\mathbb{Z}+1 & 2\mathbb{Z} \end{bmatrix}$. Computer aid largely supports to state and try to prove the following (surprising)

Conjecture. *For an integer n and units $U_n = \begin{bmatrix} -2n & -2n+1 \\ -2n-1 & -2n \end{bmatrix}$, $-(I_2 + U_n)$ is not (nontrivially) clean, for $n \leq -7$, $n = -2$ and $n \geq 4$. Equivalently, only for $n \in \{-6, -5, -4, -3, -1, 0, 1, 2, 3\}$, $-(I_2 + U_n)$ is (nontrivially) clean.*

In order to keep our exposition fluent, the results regarding this conjecture are relocated in Appendix 1. Appendix 2 contains some well-known reductions, available in Number Theory for quadratic Diophantine equations, applied to the equations associated to the nontrivially clean matrices whose negative is not nontrivially clean, and some examples.

2 Negative clean rings

To simplify the writing, $\mathrm{cn}_0(R)$ denotes *all the clean elements of R whose negative is not clean*, and $\mathrm{cn}_1(R)$ denotes *all the clean elements of R whose negative is clean*. Clearly, $\mathrm{cn}(R) = \mathrm{cn}_0(R) \cup \mathrm{cn}_1(R)$ and $\mathrm{cn}_0(R) \cap \mathrm{cn}_1(R) = \varnothing$.

With these notations, a ring is negative clean iff $-\mathrm{cn}(R) = \mathrm{cn}(R) = \mathrm{cn}_1(R)$ (or $\mathrm{cn}_0(R) = \varnothing$). Equivalently, for every pair $(e, u) \in \mathrm{Id}(R) \times U(R)$, there is a pair $(f, v) \in \mathrm{Id}(R) \times U(R)$, such that $e + u = -(f + v)$.

Since $\mathrm{cn}(R)$ is *not* in general closed under addition, $\mathrm{cn}_1(R)$ may *not* be an additive subgroup of $(R, +)$.

It is easy to give examples of clean elements whose negatives are clean.

Recall that an element a in a ring R is *strongly π-regular* if for some $n > 0$, $a^n \in a^{n+1}R \cap Ra^{n+1}$ and it is a *tripotent* if $a = a^3$ (see also [5]). If a is strongly π-regular, so is $-a$. Hence

Lemma 1. *Strongly π-regular elements (are strongly clean and) have clean negatives. In particular, tripotents and more special, idempotents and minus idempotents have clean negatives. Units, nilpotents and elements in the Jacobson radical have clean negatives.*

Therefore

$$\mathrm{Tri}(R) \cup U(R) \cup N(R) \cup J(R) \subseteq \mathrm{s}\pi\mathrm{reg}(R) \subseteq \mathrm{cn}_1(R)$$

where $\mathrm{Tri}(R)$ denotes the set of the tripotents of R and $\mathrm{s}\pi\mathrm{reg}(R)$ denotes the set of the strongly π-regular elements of R.

Notice that every strongly π-regular element is strongly clean, so all examples above are *strongly clean*. However $\mathrm{scn}(R) \nsubseteq \mathrm{cn}_1(R)$. To give an example, over a commutative ring such that 2 and 3 are not units (e.g., over $\mathbb{Z}$), consider the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = I_2 + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which is ($I_2$-)clean (and so strongly clean) but its negative is not clean. Indeed, $-A$ is not trivially clean (here $\det(-A - I_2) = 3$) and $\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1+x & y \\ z & -x \end{bmatrix}$ (the RHS is a nontrivial idempotent, so has trace 1 and zero determinant $x^2 + x + yz = 0$) has determinant $-2x$, so is not a unit whenever 2 is not a unit.

Remark. In order to give examples of negative clean rings, observe that any ring R for which the above union, or just a part of it, equals the whole ring R, gives an example of negative clean ring. As an example recall (see [11]) that for a ring R, $\mathrm{Id}(R) \cup U(R) = R$ iff R is a *division ring* or R is a *Boolean* ring. Or, if $U(R) \cup N(R) = R$ then R is *local*. Or, if $U(R) \cup \mathrm{Id}(R) \cup -\mathrm{Id}(R) = R$ for a commutative ring R (see [1]) then R is any of (1) a field, (2) a Boolean ring, (3) $\mathbb{Z}_3 \times B$ where B is a Boolean ring, or (4) $\mathbb{Z}_3 \times \mathbb{Z}_3$.

As seen above, *strongly π-regular rings are negative clean.*

Negative clean rings have different features compared to clean rings. In the sequel we mention three of these.

F1. It is well-known that a polynomial ring is *not* clean (unless over the zero ring), as the indeterminate is not clean. However, *if R is any commutative or reduced negative clean ring then $R[X]$ is also negative clean* (as for the reduced case see [13], Corollary 1.7).

However, the polynomials over a field - or a reduced or commutative ring - contain no idempotents not belonging to the base ring and therefore only clean elements of this type. For a "better" example see **F3**.

F2. Since *the polynomial rings over a nonzero exchange rings are never exchange rings* (see [10]), *negative clean rings need not be exchange* (just take any nonzero clean ring).

F3. As mentioned before, clean rings are both exchange and negative clean. In order to show that the converse fails, we can use [16] (Example 3.1, starting with a field F of characteristic 2). Indeed, there exists an exchange ring of characteristic 2 that is not clean, so it is negative clean.

F4. It is well-known that ring homomorphic images of clean rings are clean. However, ring *homomorphic images of negative clean rings need not be negative clean*. Such an example, constructed by G. Bergman, is presented here with his kind permission.

As seen before, any polynomial ring $k[X_1, X_2, ...]$ over a field k is negative clean. A homomorphic image of $k[X_1, X_2]$ is $k[X, X^{-1}]$, the ring of Laurent polynomials. In this ring, $1 + X$ is clean, since X is a unit; but if the characteristic of k is not 2, then its negative, $-1 - X$, is not clean, since the results of subtracting from $-1-X$ the idempotents, 0 and 1, are respectively $-1-X$ and $-2 - X$, neither of which is a unit (the units are aX^n for some integer n and $a \neq 0$).

For a (unital, i.e. $f(1) = 1$) ring homomorphism $f : R \to S$ we say *idempotents (or units) lift modulo* f, if for any $f(a) \in \mathrm{Id}(S)$ (resp. $f(a) \in U(S)$), there exists $e^2 = e \in R$ (resp. $u \in U(R)$) such that $f(a) = f(e)$ (resp. $f(a) = f(u)$).

Notice that if R is negative clean and idempotents and units lift modulo f (e.g., f is a retraction), then $f(R)$ is negative clean.

Proposition 2. *(i) Let* $a = e + u \in \mathrm{cn}(R)$. *If* $1 + u \in U(R)$ *then* $-a \in \mathrm{cn}(R)$ *and so* $a \in \mathrm{cn}_1(R)$.

(ii) $\mathrm{cn}(R) \cap \mathrm{ncn}(R) \subseteq \mathrm{cn}_1(R)$, *if* $\mathrm{ncn}(R)$ *denotes the nil-clean elements of* R.

(iii) If $2 \in J(R)$ *(e.g.* $2 \in N(R)$ *or* $\mathrm{char}(R) = 2$*) then* R *is negative clean.*

(iv) If I *is a nil ideal in* R *and* R *is negative clean, so is* R/I. *If* R *is not negative clean, and* I *is an ideal,* R/I *may be (negative) clean.*

(v) Direct products of rings are negative clean iff all components are negative clean.

(vi) A ring with only trivial idempotents is negative clean iff $(U(R) - 1) \cup (U(R) - 2) \subseteq \{0\} \cup U(R)$.

(vii) Subrings of (negative) clean rings need not be (negative) clean.

(viii) $\mathrm{cn}_0(R)$ *and* $\mathrm{cn}_1(R)$ *are closed under conjugations.*

Proof. (i) Indeed, $-a = -e - u = (1 - e) - (1 + u) \in \mathrm{cn}(R)$.

(ii) Indeed, if $a \in \mathrm{ncn}(R)$ then $a = e + t$ with $e^2 = e$ and $t \in N(R)$. Hence $-a = (1 - e) - (1 + t) \in \mathrm{cn}(R)$.

(iii) The inclusion $1 + J(R) \subseteq U(R)$ is well-known and it implies $U(R) + J(R) = U(R)$. Suppose $a = e+u \in \mathrm{cn}(R)$. Then $-a = -e-u = -(u+2e)+e \in \mathrm{cn}(R)$ because if $2 \in J(R)$, $u + 2e \in U(R) + J(R) = U(R)$.

(In particular, $2 = 0$ implies $e = -e$ for any idempotent and so $-(e+u) = e - u \in \mathrm{cn}(R)$).

(iv) This follows easily from the paragraph before this proposition, since idempotents and units lift modulo nil ideals.

Finally, $\mathbb{Z}$ is not negative clean, but every proper factor ring is finite, so clean and so negative clean.

(v) Obvious.

(vi) If $\mathrm{Id}(R) = \{0, 1\}$ then $\mathrm{cn}(R) = U(R) \cup (U(R) + 1)$. While $-U(R) = U(R)$, $-(U(R)+1) \subseteq U(R) \cup (U(R)+1)$ is equivalent to $(U(R)-1) \cup (U(R)-2) \subseteq \{0\} \cup U(R)$.

(vii) For example, $\mathbb{Q}$, the field of all rational numbers, is an (negative) clean ring but the subring $\mathbb{Z}$, the integer of integers, is not (negative) clean.

(viii) Since idempotents and units are invariant to conjugations, so are the clean or not clean elements. Clearly $u^{-1}(-a)u = -u^{-1}au$, for any unit u and element a of R. □

Remark. Related to (iii): Ahn, Anderson ([1], 2006) introduced (in the commutative case) the following definition: a ring R is *weakly clean* if every $a \in R$ can be written as $a = u+e$ or $a = u-e$ where $u \in U(R)$ and $e \in \mathrm{Id}(R)$.

Clearly, since negatives of units are also units, R is weakly clean iff for every $a \in R$, at least one of a or $-a$ is clean. Equivalently, $\mathrm{cn}(R) \cup (-\mathrm{cn}(R)) = R$. In the paper, it is proved that *if R is weakly clean and $2 \in J(R)$ then R is clean*.

We can slightly generalize this: *every weakly clean and negative clean ring is clean*. Indeed, in this case $\mathrm{cn}(R) \cup (-\mathrm{cn}(R)) = \mathrm{cn}(R) = R$.

While images of negative clean rings may not be negative clean, we have the following converse

Proposition 3. *Let I be an ideal of R such that $I + U(R) \subseteq U(R)$ and idempotents lift in R modulo I. If R/I is negative clean then so is R.*

Proof. Let $a = e + u$ be clean in R. Then $\overline{a}$ is clean in R/I, so by hypothesis, $-\overline{a} = \overline{f} + \overline{v}$ with idempotent $\overline{f}$ and unit $\overline{v}$ in R/I. Since idempotents lift modulo I, we may assume that $f^2 = f$ in R. We can also assume v is a unit in R. Indeed, if $\overline{vw} = \overline{wv} = \overline{1}$ then $vw = 1 + i_1$, $wv = 1 + i_2$ for some

$i_1, i_2 \in I$. By hypothesis, both vw, wv are units so both v, w are units. Therefore $-a = f + v + i$ for some $i \in I$, is clean since $v + i$ is a unit in R. $\square$

Corollary 4. *If idempotents lift in R modulo $J(R)$ and $R/J(R)$ is negative clean then R is negative clean.*

If R is a commutative ring and $\mathbb{T}_n(R)$ denotes the ring of *upper triangular* matrices over R, we can prove the following

Proposition 5. *$\mathbb{T}_n(R)$ is negative clean iff R is negative clean.*

Proof. Let $a = e + u \in \mathrm{cn}(R)$. Then $aI_n = eI_n + uI_n \in \mathrm{cn}(\mathbb{T}_n(R))$ and so $-aI_n = E + U \in \mathrm{cn}(\mathbb{T}_n(R))$, by hypothesis. Then $-a = e_{11} + u_{11} \in \mathrm{cn}(R)$. Conversely, suppose $A = E + U$ is clean in $\mathbb{T}_n(R)$. Then $a_{ii} = e_{ii} + u_{ii}$ are all clean in R for $1 \leq i \leq n$. Hence, by hypothesis, $-a_{ii} = f_i + v_i$ are also clean in R. Then $-A = \begin{bmatrix} f_1 + v_1 & -a_{12} & \cdots & -a_{1n} \\ 0 & f_2 + v_2 & \cdots & -a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & f_n + v_n \end{bmatrix} =$
$\begin{bmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & f_n \end{bmatrix} + \begin{bmatrix} v_1 & -a_{12} & \cdots & -a_{1n} \\ 0 & v_2 & \cdots & -a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & v_n \end{bmatrix} \in \mathrm{cn}(\mathbb{T}_2(R))$. $\square$

Similarly one proves

Proposition 6. *For a ring R and an R–R-bimodule M, the trivial extension $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$ is negative clean iff R is negative clean.*

Remark. Since $\mathbb{T}_n(R)$ over any ring $R \neq 0$ is not right self-injective, this also shows that *right self-injective rings need not be negative clean.*

Next, a result for some (full) matrix rings.

Following P. M. Cohn [7], a ring R is called *projective-free*, if every finitely generated projective R-module is free of unique rank. By [7] (Proposition 4.5), *a ring R is projective-free precisely when R has invariant basis number and every nontrivial idempotent matrix over R is similar to a matrix of the form* $\begin{bmatrix} I_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $1 \leq k < n$. Projective-free rings include local rings (see [7], Corollary 5.5).

Theorem 7. *Let R be a projective-free ring. $\mathbb{M}_n(R)$ is negative clean iff for every unit $U \in \mathbb{M}_n(R)$, $U - I_n$ and all differences $U - \begin{bmatrix} I_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $1 \leq k < n$, are clean.*

Proof. One way is immediate since $-U + \begin{bmatrix} I_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ are clean, for every ring R (not necessarily projective-free).

Conversely, as mentioned above, the idempotents are similar to $\begin{bmatrix} I_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. Let $A = E - U$ be clean with $E^2 = E$ and unit U.

If $E = 0_n$, $-A = U$ is clean. If $E = I_n$ then (by hypothesis), $-A = U - I_n$ is clean. Finally, if E is a nontrivial idempotent, $-A = U - E$ is clean iff so is any similar matrix. If $V^{-1}EV = \begin{bmatrix} I_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, then $-A$ is clean iff $-V^{-1}AV = V^{-1}UV - \begin{bmatrix} I_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ is clean. This is our hypothesis. □

It follows that *matrix rings over negative clean rings may not be negative clean*.

Example. Consider $\mathbb{M}_2(R[X])$, for a commutative (negative) clean ring R such that $2, 3 \notin U(R)$. As seen before, $R[X]$ is (commutative) negative clean (but not clean) and $U(R[X]) = U(R)$, $\mathrm{cn}(R[X]) = \mathrm{cn}(R)$. Take the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2(R[X])$, which is clean but $-A$ is not clean.

In [9], it was proved that split extensions of clean rings are clean. The previous example shows that an analogous result fails for negative clean rings.

Finally for formal power series we have

Proposition 8. *$R[[X]]$ is negative clean iff R is negative clean.*

Proof. Notice that $\mathrm{cn}(R[[X]]) = \mathrm{cn}(R) + XR[[X]]$. Hence $R[[X]]$ is negative clean whenever R is negative clean. Conversely, as mentioned in Proposition 2 (iii), images of negative clean rings are negative clean if idempotents and units lift modulo the given ring homomorphism. In our case, we take $\varphi : R[[X]] \to R$, $\varphi(a + bx + cx^2 + ...) = a$, a retraction, such that idempotents and units lift modulo φ. □

Regarding *corners* and *centers* of negative clean rings, two **questions** remain **open**.

Stated in [9] as an open question, it was a hard task (and took 11 years) to find a clean ring with a nonclean corner (see [16], Example 3.1).

Should this corner be also not negative clean, the same example would show that corners of negative clean rings need not be negative clean.

The final (simple) argument in the clean example is that every idempotent and every unit in this corner is upper triangular (ignoring the first finitely many rows) and so the matrices that are nonzero below the main diagonal (ignoring the first finitely many rows) cannot be written as a sum of an idempotent and a unit in the corner. Unfortunately, this does not work in the negative clean case where an upper triangular clean matrix would be necessary, whose negative is not clean. We were not able to find it.

In [4], an example of clean ring with a nonclean center was given. Should this center be also not negative clean, the same example would show that centers of negative clean rings need not be negative clean. The example relies on the property that homomorphic images of clean rings are clean. Since this property fails for negative clean rings, the example is not suitable for this purpose.

In closing this section we mention three directions for further study.

1) Are Abelian (i.e., with central idempotents) negative clean rings, clean ?

2) Clean endomorphisms of modules are characterized by the so-called ABCD-decomposition property (that is, if $\varphi \in \mathrm{End}(M_k)$ then φ is clean iff there are right k-module decompositions $M = A \oplus B = C \oplus D$ such that $\varphi|_A$ maps A isomorphically to C and $(1-\varphi)|_B$ maps B isomorphically to D). Can we find an analogue for negative clean endomorphisms of modules ?

3) Find more special properties for elements in $cn_1(R)$ (or $cn_0(R)$).

3 Clean 2×2 matrices with not clean negatives

The goal of this section is to describe, as much as possible, *the clean* 2×2 *integral matrices whose negative is (not) clean*.

Notice that if the negative of a clean matrix is not clean, so are its transpose and the matrix obtained by interchanging the (main) diagonal entries.

To see the latter, conjugation by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and transpose will do.

We also mention that many results in this section hold in more general conditions, that is, commutative rings and/or GCD domains (a domain is *GCD* if every two elements have a greatest common divisor).

Since nontrivially clean 2×2 integral matrices are characterized by systems of equations, we have to deal separately with I_2-clean matrices and with nontrivially clean matrices.

More precisely, we have to answer four questions

(A) which are the I_2-clean matrices whose negative is not I_2-clean;
(B) which are the I_2-clean matrices whose negative is not nontrivially clean;
(C) which are the nontrivially clean matrices whose negative is not I_2-clean;
(D) which are the nontrivially clean matrices whose negative is not nontrivially clean.

Recall that nontrivially clean matrices are characterized by

Theorem 9. *A 2×2 integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nontrivially clean iff any of the systems*

$$\left\{ \begin{array}{ll} x^2 + x + yz = 0 & (1) \\ (a-d)x + cy + bz + \det(A) - d = \pm 1 & (\pm 2_A) \end{array} \right.$$

with unknowns x, y, z, has at least one solution over $\mathbb{Z}$. If $b \neq 0$ and any of $(\pm 2_A)$ holds, then (1) is respectively equivalent to

$$bx^2 - (a-d)xy - cy^2 + bx + (d - \det(A) \pm 1)y = 0 \qquad (\pm 3_A).$$

Corollary 10. *If $\det(A) = 0$ then A is nontrivially clean iff $-A$ is nontrivially clean.*

Proof. Indeed, since $\det(A) = \det(-A)$, if both are zero, the systems (1), $(\pm 2_A)$ and (1), $(\pm 2_{-A})$ coincide. $\square$

3.1 Diagonal and zero second row matrices

We have already seen that the negative of a clean diagonal matrix may not be clean: $2I_2 = I_2 + I_2$ is clean but $-2I_2$ is not clean. However we can describe the diagonal matrices in $\mathrm{cn}_0(\mathbb{M}_2(\mathbb{Z}))$.

Proposition 11. *Let $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ be clean. Then $-A$ is not clean iff*

(i) $a = d = 2$;
(ii) $a-d$ is odd, $d(d-1) \equiv \pm 1$ (mod$(a-d)$) but $d(d+1) \not\equiv \pm 1$ (mod$(a-d)$).

Proof. This follows immediately from Theorem 4, [6]. $\square$

Example. $\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix} + \begin{bmatrix} 7 & -2 \\ 3 & -1 \end{bmatrix}$ is clean but $-\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$ is not clean ($2 \cdot 3 \not\equiv \pm 1$ (mod 3).

Clean integral matrices with zero (second) row (see [12]) provide some examples of clean matrices with not clean negative.

Proposition 12. *Let* $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ *be a clean integral matrix. Its opposite* $-A$ *is also clean iff* $a \neq 2$, *or,* $a = 2$ *and* b *is odd.*

Proof. Having zero determinant, the matrix $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ is not 0_2-clean and is I_2-clean iff $a \in \{0, 2\}$ and arbitrary b. If $a = 0$, $-A = \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix}$ is nilpotent so clean for any b, and, for $a = 2$, the matrices $-A = \begin{bmatrix} -2 & -b \\ 0 & 0 \end{bmatrix}$ are clean iff b is odd. Since $\det(A) = 0$ we just use Corollary 10 in the nontrivially clean case. □

3.2 I_2-clean integral matrices

Proposition 13. ***(A)*** *For a unit* U, *an* I_2*-clean matrix* $I_2 + U$ *has an* I_2*-clean negative iff any of the following holds*

(i) $\mathrm{Tr}(U) = -3$ *and* $\det(U) = 1$,

(ii) $\mathrm{Tr}(U) = -2$, *and* $\det(U) \in \{\pm 1\}$,

(iii) $\mathrm{Tr}(U) = -1$ *and* $\det(U) = -1$.

Proof. Let $U = [u_{ij}]$ be any 2×2 unit. The I_2-clean matrix $I_2 + U$ has an I_2-clean negative iff $-I_2 - U = I_2 - V$ with a unit V, that is, $V = U + 2I_2$.

Since $\det(U + 2I_2) = \det(U) + 2\mathrm{Tr}(U) + 4$ and $\det(U) \in \{\pm 1\}$, clearly, V is not a unit if $\mathrm{Tr}(U) \notin \{-3, -2, -1\}$ and is a unit in the cases stated. □

Remark. The units in the above proposition are: (i) $\begin{bmatrix} a-3 & b \\ c & -a \end{bmatrix}$ with $a(a-3) + bc = -1$, (ii) $\begin{bmatrix} a-2 & b \\ c & -a \end{bmatrix}$ with $a(a-2) + bc \in \{\pm 1\}$ and (iii) $\begin{bmatrix} a-1 & b \\ c & -a \end{bmatrix}$ with $a(a-1) + bc = 1$.

Proposition 14. ***(B)*** *Let* $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a unit with* $b \neq 0$ *and* $\det(U) = ad - bc = 1$. *Then* $-A = -(I_2 + U)$ *is nontrivially clean iff the system*

$$\begin{array}{c} bx^2 - (a-d)xy - cy^2 + bx + (a + 2d + 3 \pm 1)y = 0 \qquad (\pm 3_{-A}) \\ (a-d)x + cy + bz - a - 2d - 3 = \pm 1 \qquad (\pm 2_{-A}) \\ ad - bc = 1 \end{array} .$$

has (integer) solutions for the unknowns x, y, z.

Proof. Using Theorem 9, for the cleanness of $-A$ we reach the system

$$\begin{array}{cc} bx^2-(a-d)xy-cy^2+bx+(d+1+\det(A)\pm 1)y=0 & (\pm 3_{-A}) \\ (a-d)x+cy+bz-\det(A)-d-1=\pm 1 & (\pm 2_{-A}) \end{array}.$$

Since $\det(A)=\mathrm{Tr}(U)+\det(U)+1=\mathrm{Tr}(U)+2=a+d+2$ we can rewrite the system as stated. $\square$

Corollary 15. *For any unit* $U\in\begin{bmatrix} 2\mathbb{Z}+1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z}+1 \end{bmatrix}$, $-(I_2+U)$ *is not (nontrivially) clean.*

Proof. Indeed, $\pm A:=\pm(I_2+U)\in\mathbb{M}_2(2\mathbb{Z})$ and so the equations $(\pm 2_{-A})$ reduce to $2k=\pm 1$, for some k, with no integer solutions. Hence A is I_2-clean and $-A$ is not nontrivially clean. $\square$

Remarks. 1) As mentioned in the Introduction, *the negative of a strongly (or uniquely, or both strongly and uniquely) clean element need not be clean.* An example deduced from the above corollary is the both strongly and uniquely clean matrix $A=\begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix}=I_2+\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$. Indeed, $-A$ is not trivially clean nor nontrivially clean since $(\pm 2_{-A})$ are $2(2x+y-z+4)=\pm 1$, with no integer solutions.

A is obviously strongly clean and it is also uniquely clean since it is not 0_2-clean nor nontrivial clean: $(\pm 2_A)$ are $2(-2x-y+z)=\pm 1$, with no integer solutions.

2) If for a unit $U=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we take $b\in\{\pm 1\}$, then $(\pm 2_{-A})$ have always solution for z for any given x, y. Hence, since $(\pm 3_{-A})$ have always at least the solutions $(0,0)$ and $(-1,0)$, it follows that $-(I_2+U)$ is nontrivial clean. Actually, more generally, if an entry on the secondary diagonal is a unit, the matrix is clean.

3.3 A conjecture

Notice that any unit in $\mathbb{M}_2(\mathbb{Z})$ may have only one or two even entries (0 or 3 or 4 even entries yield even determinant). One, in any position, but two, only on diagonals. Excepting the matrices in the previous corollary, we can consider the units $U\in\begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z}+1 \\ 2\mathbb{Z}+1 & 2\mathbb{Z} \end{bmatrix}$.

For such matrices the above corollary fails. For the unit $U=\begin{bmatrix} -2 & -3 \\ -5 & -8 \end{bmatrix}$,

$A = I_2 + U$ is I_2-clean and $-A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}$ is nontrivially clean.

It is difficult to prove results starting with such units. Using computer aid we were able to state the following

Conjecture 16. *For an integer n and units $U_n = \begin{bmatrix} -2n & -2n+1 \\ -2n-1 & -2n \end{bmatrix}$, $-(I_2+U_n)$ is not (nontrivially) clean, for $n \leq -7$, $n = -2$ and $n \geq 4$. Equivalently, only for $n \in \{-6, -5, -4, -3, -1, 0, 1, 2, 3\}$, $-(I_2+U_n)$ is (nontrivially) clean.*

The proof of this statement reduces to some hard to solve quadratic Diophantine equations with parametric coefficients. The partial results we were able to obtain are given in Appendix 1.

3.4 Nontrivially clean matrices

Proposition 17. ***(C)*** *For a nontrivially clean matrix A, the negative $-A$ is not I_2-clean iff* $\det(A) + \mathrm{Tr}(A) + 1 \notin \{\pm 1\}$.

Proof. $-A$ is not I_2-clean iff $-I_2 - A$, or equivalently, $I_2 + A$ is not a unit. Indeed, this holds iff $\det(I_2 + A) = \det(A) + \mathrm{Tr}(A) + 1 \notin \{\pm 1\}$. $\square$

Example. $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = E_{11} + I_2$ is clean, but not trivially clean (both A and $A - I_2$ are not units). Since $\det(A) + \mathrm{Tr}(A) + 1 = 5$, $-A$ is not I_2-clean. However, it is clean: $-A = \begin{bmatrix} 3 & 2 \\ -3 & -2 \end{bmatrix} + \begin{bmatrix} -5 & -2 \\ 3 & 1 \end{bmatrix}$.

3.5 The D case

According to Proposition 2, (ii), since *every nontrivial idempotent is similar to E_{11}*, up to similarity, it suffices to characterize the E_{11}-clean 2×2 integral matrices whose opposite is *not* nontrivially clean.

Therefore we can start with any unit $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and consider $A = E_{11} + U$, respectively $-A = -E_{11} - U$. Since $-U$ is also a unit, we just have to characterize *the units U such that $B = U - E_{11}$ is not nontrivially clean.* Observe that $\det(B) = \det(U) - d = \pm 1 - d$, so that using Theorem 9, we obtain

Proposition 18. *Suppose* $\det(U) = ad - bc = 1$. *Then* $U - E_{11}$ *is nontrivially clean iff the system*

$$\begin{gathered} bx^2 - (a-d-1)xy - cy^2 + bx + (2d-1\pm 1)y = 0 \qquad (\pm 3_{U-E_{11}}) \\ (a-d-1)x + cy + bz = 2d - 1 \pm 1 \qquad (\pm 2_{U-E_{11}}) \\ ad - bc = 1 \end{gathered}$$

has integer solutions.

Proposition 19. *Suppose* $\det(U) = ad - bc = -1$. *Then* $U - E_{11}$ *is nontrivially clean iff the system*

$$\begin{gathered} bx^2 - (a-d-1)xy - cy^2 + bx + (2d+1\pm 1)y = 0 \qquad (\pm 3_{U-E_{11}}) \\ (a-d-1)x + cy + bz = 2d + 1 \pm 1 \qquad (\pm 2_{U-E_{11}}) \\ ad - bc = -1 \end{gathered}$$

has integer solutions.

It is easy to discard the trivially clean cases.

Lemma 20. *If* $\det(U) = 1$, $U - E_{11}$ *is* 0_2*-clean iff* $d \in \{0, 2\}$ *and* I_2*-clean iff* $a + 2d \in \{2, 4\}$. *If* $\det(U) = -1$, $U - E_{11}$ *is* 0_2*-clean iff* $d \in \{-2, 0\}$ *and* I_2*-clean iff* $a + 2d \in \{0, 2\}$.

Also recall that if $b \in \{\pm 1\}$ or $c \in \{\pm 1\}$ then $U - E_{11}$ is clean.

Using the previous propositions, it is easy to check

Lemma 21. *All* 2×2 *(integral) units* U *with one or two zeros, yield clean matrices* $U - E_{11}$.

Proof. The two zeros on the diagonal case is easy. As for the *upper triangular* units (including, two zeros on the secondary diagonal), notice that for units of form $U = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, $U - E_{11}$ is idempotent, so clean. For units of form $\begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix}$, we get clean decompositions

$$\begin{bmatrix} -2 & b \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3-2b & 2b^2-5b+3 \\ -2 & 2b-2 \end{bmatrix} + \begin{bmatrix} 2b-5 & -2b^2+6b-3 \\ 2 & 1-2b \end{bmatrix}.$$

Analogously for $\begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}$. The remaining cases: (1,1) or (2,2) zero entry, are also easy. The lower triangular case is analogous. □

Lemma 22. *All units* $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *with* $\det(U) = d$, *yield clean matrices* $U - E_{11}$.

Proof. If $\det(U) = d$ then $\det(U - E_{11}) = 0$. Since $E_{11} - U$ is always clean, so is $U - E_{11}$, by Corollary 10. $\square$

Therefore, it only remains to describe *the units U without zero entries* (and with $b, c \notin \{\pm 1\}$).

Example. For $U_{-2} = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$, we have proved in case **(B)** that $-I_2 - U_{-2}$ is not clean. However, $U_{-2} - E_{11} = \begin{bmatrix} 3 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 8 \\ -1 & 7 \end{bmatrix}$ is clean.

It was easy to write a code which, given a unit U, subtracts a nontrivial idempotent $\begin{bmatrix} x+1 & y \\ z & -x \end{bmatrix}$ (with $x^2 + x + yz = 0$) and checks whether this difference has determinant ± 1. With the entries x, y, z bounded by 100, the computer provided some 16,700 candidates of units U such that $U - E_{11}$ is not nontrivially clean.

We were able to prove the following results.

Proposition 23. *(a) Consider the units* $V_n = \begin{bmatrix} 3n-1 & 3n \\ 3n & 3n+1 \end{bmatrix}$, *for any integer* $n \neq -1$. *Then* $V_n - E_{11}$ *is not clean.*

(b) Consider the units $W_b = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ *with* $b \notin \{\pm 1\}$, $d \notin \{0, 2\}$, $a + 2d \notin \{2, 4\}$ *and* $ad = b^2 - 1$ *(i.e.,* $\det(W_b) = -1$ *and both* $W_b - E_{11}$, $W_b - E_{11} - I_2$ *are not units). If* b *divides* $a - d - 1$ *and* $2d + 1$ *then* $W_b - E_{11}$ *is not clean.*

The case $ad = b^2 + 1$ *(i.e.,* $\det(W_b) = 1$*), and the case* $c = -b$ *are analogous.*

(c) Consider the units $W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *such that none of* $W - E_{11}$, $W - E_{11} - I_2$ *is a unit and* $\delta = \gcd(b; c) \neq 1$. *If* δ *divides* $a - d - 1$ *and* $2d + 1$ *then* $W - E_{11}$ *is not clean.*

Proof. (a) Since for $n \neq -1$, $\det(V_n - E_{11}) = -(3n + 2) \notin \{\pm 1\}$ and $\det(V_n - E_{11} - I_2) = -9n \notin \{\pm 1\}$, the matrices $V_n - E_{11}$ are not trivially clean. These are also not nontrivially clean since for these (± 2) is $3(-x + ny + nz - 2n - 1) = \pm 1$, with no integer solutions.

(b) The condition $d \notin \{0, 2\}$ assures $W_b - E_{11}$ is not a unit, and the condition $a + 2d \notin \{0, 2\}$ assures $W_b - E_{11} - I_2$ is not a unit. In the remaining case, (± 2) is $(a - d - 1)x + by + bz - (2d - 1) = \pm 1$, with no integer solutions since $b \notin \{\pm 1\}$ divides $a - d - 1$ and $2d + 1$.

(c) Analogous with case (b). $\square$

Remark. One could argue that requiring $c \in \{\pm b\}$ is already a very strong condition. Actually, it is not: out of 282 units U with $U - E_{11}$ not nontrivially clean (for entries $x, y, z \leq 17$), 122 have $c \in \{\pm b\}$, that is, more than 43% !

More details and examples on the (D) case appear in Appendix 2.

4 Appendix 1

This section includes our results regarding the Conjecture 16, which we recall as

Theorem 24. *For an integer n and units $U_n = \begin{bmatrix} -2n & -2n+1 \\ -2n-1 & -2n \end{bmatrix}$, $-(I_2 + U_n)$ is not (nontrivially) clean, for $n \leq -7$, $n = -2$ and $n \geq 4$. Equivalently, only for $n \in \{-6, -5, -4, -3, -1, 0, 1, 2, 3\}$, $-(I_2 + U_n)$ is (nontrivially) clean.*

The $n \in \{-6, -5, -4, -3, -1, 0, 1, 2, 3\}$ cases are covered by simply giving clean decompositions for these $A_n := -(I_2 + U_n) = \begin{bmatrix} 2n-1 & 2n-1 \\ 2n+1 & 2n-1 \end{bmatrix}$.

$n = -6$: $\begin{bmatrix} -13 & -13 \\ -11 & -13 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} -17 & -19 \\ -9 & -10 \end{bmatrix}$;

$n = -5$: $\begin{bmatrix} -11 & -11 \\ -9 & -11 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} -15 & -17 \\ -7 & -8 \end{bmatrix}$;

$n = -4$: $\begin{bmatrix} -9 & -9 \\ -7 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -10 & -13 \\ -7 & -9 \end{bmatrix}$;

$n = -3$: $\begin{bmatrix} -7 & -7 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -8 & -11 \\ -5 & -7 \end{bmatrix}$;

$n = -1$: $\begin{bmatrix} -3 & -3 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -4 & -11 \\ -1 & -3 \end{bmatrix}$;

$n = 0$: $\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}$;

$n = 1$: $\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$;

$n = 2$: $\begin{bmatrix} 3 & 3 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$;

$n = 3$: $\begin{bmatrix} 5 & 5 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix}$.

Next, observe that $\det(A_n) = -2(2n-1)$, so A_n is not a unit, and so is $A_n - I_2$ since $\det(A_n - I_2) = \det(A_n) - \mathrm{Tr}(A_n) + 1 = -4(2n-1) + 1 = -8n + 5$

(because $-8n+5 = \pm 1$ has no integer solutions). Hence all A_n are *not trivially* clean.

Therefore, the proof of this theorem reduces to

Theorem 25. *For integers n, the matrices* $\begin{bmatrix} 2n-1 & 2n-1 \\ 2n+1 & 2n-1 \end{bmatrix}$ *are not nontrivially clean for $n \leq -7$, $n = -2$ and $n \geq 4$.*

We further reduce the proofs of these theorems to the following

Theorem 26. *For any integer $n \notin \{-6, -5, -4, -3, -1, 0, 1, 2, 3\}$, consider the parametric Pell equations*

$$\begin{array}{ll} (+) & s^2 - (4n^2-1)t^2 = 32n^2 - 24n + 5 \\ (-) & s^2 - (4n^2-1)t^2 = 32n^2 - 48n + 17 \end{array}$$

with unknowns s, t.

For any solution s of the $(+)$ equation, denote $k^+_{1,2} = \dfrac{12n^3 - 10n^2 - n + 1 \pm s}{4n^2-1}$, *and for any solution s of the $(-)$ equation denote* $k^-_{1,2} = \dfrac{12n^3 - 14n^2 + n + 2 \pm s}{4n^2-1}$. *Then none of these four k's is an integer.*

Proof. Using Theorem 9 for nontrivially clean 2×2 integral matrices, we have to show that *none* of the systems

$$(+_1) \quad \begin{array}{c} (2n-1)x^2 - (2n+1)y^2 + (2n-1)x + 2(3n-1)y = 0 \\ (2n+1)y + (2n-1)z - 2(3n-1) = 0 \end{array}$$

$$(-_1) \quad \begin{array}{c} (2n-1)x^2 - (2n+1)y^2 + (2n-1)x + 2(3n-2)y = 0 \\ (2n+1)y + (2n-1)z - 2(3n-2) = 0 \end{array}$$

has integer solutions for $n \notin \{-6, -5, -4, -3, -1, 0, 1, 2, 3\}$.

We first refer to the linear Diophantine equations.

Since

$$(2n+1)(1-n) + (2n-1)n = 1$$

we deduce that $2n+1$, $2n-1$ are coprime, so both linear equations have solutions, namely,

$(y, z) = (2(3n-1)(1-n) + k(2n-1), 2(3n-1)n - k(2n+1))$ for some $k \in \mathbb{Z}$, for the $(+)$ equation, and

$(y, z) = (2(3n-2)(1-n) + k(2n-1), 2(3n-2)n - k(2n+1))$ for some $k \in \mathbb{Z}$, for the (-) equation.

We have to show that, excepting the values in the hypothesis, the quadratic (+) equation has no integer solutions for $y = 2(3n-1)(1-n)+k(2n-1)$, and the quadratic (-) equation has no integer solution for $2(3n-2)(1-n)+k(2n-1)$.

We view both quadratic equations as degree 2 equations in x, that is,

$$(+)\quad (2n-1)x^2 + (2n-1)x - (2n+1)y^2 + 2(3n-1)y = 0$$

and

$$(\text{-})\quad (2n-1)x^2 + (2n-1)x - (2n+1)y^2 + 2(3n-2)y = 0.$$

Computing $\Delta_x^+ = (2n-1)^2 + 4(2n-1)[(2n+1)y^2 - 2(3n-1)y]$ for $y = 2(3n-1)(1-n) + k(2n-1)$ we obtain

$$\Delta_x^+ = (2n-1)^2[4(4n^2-1)k^2 - 8(12n^3-10n^2-n+1)k + 16n(9n^3-15n^2+7n-1)+1],$$

respectively,

$$\Delta_x^- = (2n-1)^2[4(4n^2-1)k^2 - 8(12n^3-14n^2+n+2)k + 16n(9n^3-21n^2+16n-4)+1],$$

computed for $y = 2(3n-2)(1-n) + k(2n-1)$.

We solve $\dfrac{\Delta_x^\pm}{(2n-1)^2} = t^2$ as degree two equations in k, which gives

$$\Delta_k^+ = 16(32n^2 - 24n + 5 + (4n^2-1)t^2)$$

and

$$\Delta_k^- = 16(32n^2 - 48n + 17 + (4n^2-1)t^2).$$

For both $32n^2 - 24n + 5 + (4n^2-1)t^2$ and $32n^2 - 48n + 17 + (4n^2-1)t^2$ there exist integers t such that these are squares.

If $F + G\sqrt{4n^2-1} = \pm(2|n| + 1\sqrt{4n^2-1})^m$ then $32n^2 - 24n + 5 + (4n^2-1) = 36n^2 - 24n + 4 = [2(3n-1)]^2$ and so at least

1. For $32n^2 - 24n + 5 + (4n^2-1)t^2 = s^2$, solutions are $(t, s) = ((\pm 1)F + (2(3n-1))G, (2(3n-1))F + (\pm(4n^2-1))G)$

and

2. For $32n^2 - 48n + 17 + (4n^2-1)t^2 = s^2$, solutions are $(t, s) = ((\pm 1)F + (2(3n-2))G, (2(3n-2))F + (\pm(4n^2-1))G)$.

The *resolvent* equation $u^2 - (4n^2-1)v^2 = 1$ has the least positive solution $(2|n|, 1)$ and the only solutions $(u, v) = (G, F)$ with $F + G\sqrt{4n^2-1} = \pm(2|n| + 1\sqrt{4n^2-1})^m$. As for the roots $k_{1,2}$ we have in the (+) case,

$$k_{1,2}^+ = \frac{4(12n^3 - 10n^2 - n + 1) \pm 4s}{4(4n^2-1)} = \frac{12n^3 - 10n^2 - n + 1 \pm s}{4n^2-1}$$

and, in the (-) case,

$$k^-_{1,2} = \frac{4(12n^3 - 14n^2 + n + 2) \pm 4s}{4(4n^2 - 1)} = \frac{12n^3 - 14n^2 + n + 2 \pm s}{4n^2 - 1},$$

which completes the proof. $\square$

Both Pell equations in the previous statement belong to the hyperbolic case. Hence these have finitely many solution classes. We succeeded proving the following special case

Theorem 27. *The hyperbolic Pell equations* $(+)$ *and* $(-)$ *are always solvable. For any integer* $n \notin \{-6, -5, -4, -3, -1, 0, 1, 2, 3\}$, *if the Pell equations have only two class solutions, then the corresponding* k*'s are not integers.*

Proof. It is readily seen that $(t, s) = (1, 2(3n - 1))$, is a fundamental solution for $(+)$ and $(t, s) = (1, 2(3n - 2))$ is a fundamental solution for $(-)$.

Moreover, if these Pell equations have only two solution classes, $(1, 2(3n - 1))$ and $(-1, 2(3n-1))$ are the fundamental solutions for $(+)$ and $(1, 2(3n-2))$ and $(-1, 2(3n-2))$ are the fundamental solutions for $(-)$. The k's corresponding to these solutions are

$$k^+_{1,2} = \frac{12n^3 - 10n^2 - n + 1 \pm 2(3n - 1)}{4n^2 - 1} = \begin{cases} \dfrac{12n^3 - 10n^2 + 5n - 1}{4n^2 - 1} \\ \dfrac{12n^3 - 10n^2 - 7n + 3}{4n^2 - 1} \end{cases} =$$

$$= \begin{cases} 3n - \dfrac{5}{2} + \dfrac{8n - \frac{7}{2}}{4n^2 - 1} \\ 3n - \dfrac{5}{2} + \dfrac{-4n + \frac{1}{2}}{4n^2 - 1} \end{cases} \text{and}$$

$$k^-_{1,2} = \frac{12n^3 - 14n^2 + n + 2 \pm 2(3n - 2)}{4n^2 - 1} = \begin{cases} \dfrac{12n^3 - 14n^2 + 7n - 2}{4n^2 - 1} \\ \dfrac{12n^3 - 14n^2 - 5n + 6}{4n^2 - 1} \end{cases} =$$

$$= \begin{cases} 3n - \dfrac{7}{2} + \dfrac{10n - \frac{11}{2}}{4n^2 - 1} \\ 3n - \dfrac{7}{2} + \dfrac{-2n + \frac{5}{2}}{4n^2 - 1} \end{cases}.$$

In general, if (F, G) is given by $F + G\sqrt{4n^2 - 1} = (2n + \sqrt{4n^2 - 1})^m$, and so $F = (2n)^m + C^2_m(2n)^{m-2}(4n^2 - 1) + C^4_m(2n)^{m-4}(4n^2 - 1)^2 + ...$, $G = C^1_m(2n)^{m-1} + C^3_m(2n)^{m-3}(4n^2 - 1) + C^5_m(2n)^{m-5}(4n^2 - 1)^2 + ...$, we have to replace in the k's formulas given in the previous theorem, $s = (2(3n-1))F + (\pm(4n^2-1))G$), for the k^+'s, and $s = (2(3n-2))F + (\pm(4n^2-1))G)$ for the k^-'s.

For the k^+'s, since $s = 2(3n-1)(2n)^m + (4n^2-1)q$ for some integer q, we just have to check when $k^+_{1,2} = \dfrac{12n^3 - 10n^2 - n + 1 \pm 2(3n-1)(2n)^m}{4n^2-1}$ is an integer. Finally, since $(2n)^2 = (4n^2-1)+1$, we have to check this by replacing $(2n)^m$ with 1 or $2n$.

For the k^-'s, we deal with $k^-_{1,2} = \dfrac{12n^3 - 10n^2 - n + 1 \pm 2(3n-2)(2n)^m}{4n^2-1}$ being an integer, and, as in the previous case, it suffices to check this by replacing $(2n)^m$ with 1 or $2n$.

When checking whether the k's are integers *with* $(2n)^m$ *replaced by* 1, it suffices to check when $4n^2-1$ divides any of $16n-7$, $1-8n$, $20n-11$ or $5-4n$. Clearly this cannot happen for large n.

More precisely, $\dfrac{|16n-7|}{4n^2-1} < 1$ iff $n \leq -5$ and $n \geq 4$; $\dfrac{|8n-1|}{4n^2-1} < 1$ iff $n \leq -3$ and $n \geq 3$; $\dfrac{|20n-11|}{4n^2-1} < 1$ iff $n \leq -6$ and $n \geq 5$ (here 4,5 are outside the "positive" area); $\dfrac{|4n-5|}{4n^2-1} < 1$ iff $n \leq -2$ and $n \geq 1$.

Hence for $n \notin \{-6,-5,-4,-3,-1,0,1,2,3\}$, no k's corresponding to the solutions $(\pm 1, 2(3n-1))$ and $(\pm 1, 2(3n-2))$ respectively, are integers, excepting $k^- = 3n - \dfrac{7}{2} + \dfrac{10n - \frac{11}{2}}{4n^2-1}$ which could be integer only for $n \in \{4,5\}$. However, it is *not* because 63 does not divide 69, nor 89.

The *replacement of* $(2n)^m$ *by* $2n$ amounts to

$$k^+_{1,2} = \frac{12n^3 - 10n^2 - n + 1 \pm 4n(3n-1)}{4n^2-1} = \begin{cases} \dfrac{12n^3 + 2n^2 - 3n + 1}{4n^2-1} \\ \dfrac{12n^3 - 22n^2 + 5n + 1}{4n^2-1} \end{cases} =$$

$$= \begin{cases} 3n + \dfrac{1}{2} + \dfrac{\frac{3}{2}}{4n^2-1} \\ 3n - \dfrac{11}{2} + \dfrac{8n - \frac{9}{2}}{4n^2-1} \end{cases}, \text{ and}$$

$$k^-_{1,2} = \frac{12n^3 - 14n^2 + n + 2 \pm 4n(3n-2)}{4n^2-1} = \begin{cases} \dfrac{12n^3 - 2n^2 - 7n + 2}{4n^2-1} \\ \dfrac{12n^3 - 26n^2 + 9n + 2}{4n^2-1} \end{cases} =$$

$$= \begin{cases} 3n - \dfrac{1}{2} + \dfrac{-4n + \frac{3}{2}}{4n^2 - 1} \\ 3n - \dfrac{13}{2} + \dfrac{12n - \frac{9}{2}}{4n^2 - 1} \end{cases}$$ and we finally have to check when $4n^2 - 1$ divides any of 3, $16n - 9$, $8n - 3$, $24n - 9$. As in the first replacement ($(2n)^m$ by 1), comparing fractions with 1, we obtain the range $n \leq -7$, $n \geq 3$ with three exceptions: $\frac{16n - 9}{4n^2 - 1}$ for $n = 4$ and $\frac{24n - 9}{4n^2 - 1}$ for $n \in \{4, 5\}$. However, 63 does not divide 55 nor 87 and 99 does not divide 111.

All the above, *completely* covers the two *class solutions* whose fundamental solutions are $(1, 2(3n-1))$, $(-1, 2(3n-1))$ in the $(+)$ case, and by $(1, 2(3n-2))$, $(-1, 2(3n-2))$ in the $(-)$ case. □

Final comments.

A) If we get an integer k such that t is an integer, there are no problems in "coming back" to (integer) x and y.

Indeed, $y = 2(3n-1)(1-n)+k(2n-1)$ or else $y = 2(3n-2)(1-n)+k(2n-1)$, and for $x_{1,2} = \frac{-(2n-1) \pm \sqrt{\Delta_x}}{2(2n-1)} = \frac{-1 \pm t}{2}$, it remains to notice that, in both $(\pm)$ cases, t^2 and so t is odd.

B) What remains uncovered is the answers to the following questions;

How many other solution classes has each of the equations $32n^2 - 24n + 5 + (4n^2 - 1)t^2 = s^2$ and $32n^2 - 48n + 17 + (4n^2 - 1)t^2 = s^2$?

Which are their fundamental solutions ?

Notice that if $(\alpha, \beta^{\pm})$ is a fundamental solution of any (other) solution class K for the $(\pm)$ equations, we just have to perform the same verification as above, for $k^{\pm}_{1,2} = \frac{12n^3 - 10n^2 - n + 1 \pm \beta^{\pm}}{4n^2 - 1}$ and $k^{\pm}_{1,2} = \frac{12n^3 - 10n^2 - n + 1 \pm 2n\beta^{\pm}}{4n^2 - 1}$, respectively.

Notice that *3 solution classes* is also possible for these Pell equations: the fundamental solutions are $(0, \sqrt{N})$ and $(\pm 1, 2(3n-1))$, $(\pm 1, 2(3n-2))$, respectively. Unfortunately, there is no integer formula for $\sqrt{N}$. When $32n^2 - 48n + 17 = q^2$, there is a formula for q of form $q = A\alpha^n + B\beta^n + C$, where A, B, C, α, and β are algebraic numbers, but not necessarily rational or integral (this follows from the general theory of linear recurrence relations).

C) We can still develop our information on the solutions of the Pell equations $(+)$ and $(-)$, by using the following recent characterization obtained in [15].

Theorem 28. *Suppose (x_1, y_1) is the least positive solution of Pell equation $x^2 - Dy^2 = 4$ and let*

$$(V,U) = \begin{cases} (\sqrt{AN(x_1-2)/D}, \sqrt{AN(x_1+2)/D} & \text{if} \quad N > 0 \\ (\sqrt{A\left|N\right|(x_1+2)/D}, \sqrt{A\left|N\right|(x_1-2)/D}) & \text{if} \quad N < 0 \end{cases}.$$

(a) *If $N > 0$, then an integer pair (u, v) satisfying $Au^2 + Buv + Cv^2 = N$, with $D = B^2 - 4AC$, is a fundamental solution iff one of the following holds:*
(i) $0 < v < V$.
(ii) $v = 0$ and $u = \sqrt{N/A}$.
(iii) $v = V$ and $u = (U - BV)/(2A)$.
(b) *If $N < 0$, then an integer pair (u, v) satisfying $Au^2 + Buv + Cv^2 = N$, is a fundamental solution iff one of the following holds:*
(i) $\sqrt{4A\left|N\right|/D} \le v < V$.
(ii) $v = V$ and $u = (U - BV)/(2A)$.

In our case, $D = 4(4n^2 - 1)$ and so $(4n, 1)$ is the least positive solution of Pell equation $x^2 - 4(4n^2 - 1)y^2 = 4$. For both equations $A = 1$, $B = 0$, $C = -(4n^2 - 1)$

Notice that for both possible (positive) N, $U = \sqrt{\dfrac{N}{2(2n-1)}}$. Since both N are odd, $U \notin \mathbb{Z}$ and $\dfrac{U - BV}{2A} = \dfrac{U}{2} \notin \mathbb{Z}$. Therefore, case (iii) in the previous theorem cannot happen.

Moreover, $V = \sqrt{\dfrac{N}{D}(4n-2)} = \sqrt{\dfrac{N}{2(2n+1)}}$.

Hence:

1. For the (+) equation, since N *is no square* ($N \equiv 5 \bmod 8$), only the (i) case, in the previous theorem can happen. Here $V = \sqrt{\dfrac{N}{2(2n+1)}} = \sqrt{8n - 10 + \dfrac{25}{2(2n+1)}}$.

Unfortunately, in the (i) case, $0 < v < \sqrt{8n - 10 + \dfrac{25}{2(2n+1)}}$ permits no conclusion on the possible fundamental solutions or on the number of solution classes.

2. For the (−) equation, $N = 32n^2 - 48n + 17$ *can be a square.* The equation $32x^2 - 48x + 17 = y^2$ has fundamental solutions $(1, 1)$ and $(2, 7)$ together with recurrences $x_{n+1} = 17x_n + 3y_n - 12$, $y_{n+1} = 96x_n + 17y_n - 72$ (see [14]).

As examples, for $(x, y) = (1, 1)$ we get $(8, 41)$, for $(x, y) = (2, 7)$ we get $(43, 239)$ and for $(8, 41)$ we get $(247, 1393)$.

Therefore, the case (ii) ($v = 0$, $u = \sqrt{N}$) in the previous theorem, can happen for the $(-)$ equation. Here $V = \sqrt{\frac{N}{2(2n+1)}} = \sqrt{8n - 16 + \frac{49}{2(2n+1)}}$.

As in the $(+)$ case, in the (i) case, $0 < v < \sqrt{8n - 16 + \frac{49}{2(2n+1)}}$ permits no conclusion on the possible fundamental solutions or on the number of solution classes.

According to quadratic Diophantine equations experts, (so far) we also cannot answer (excepting brute force over a range of n) the question:

What are the values of n for which (any of) these two equations has precisely *two* solution classes (and so exactly with fundamental solutions $(\pm 1, 2(3n-1))$ and $(\pm 1, 2(3n-2))$, respectively) ?

D) We checked these equations for $n \in \{-23, -22, ..., -1, 1, ..., 23, 24\}$ using [14].

We have 3 solution classes only for $n \in \{2, 8\}$, and this happens whenever $32n^2 - 48n + 17 = q^2$ is a square (and then $(\pm q, 0)$ is a solution for the $(-)$ equation).

For $n = 1$ we have only 1 solution class.

We have 4 solution classes for $n \in \{-17, -11, -10, -8, -5, 3, 17, 20\}$ and for all the other 36 values we have 2 solution classes. We have also 2 solution classes for $n \in \{\pm 31, \pm 40, \pm 50, \pm 60, \pm 70, \pm 80, \pm 90, \pm 100\}$). Hence, at least to this extent, there are no more that 4 solution classes and the 2 solution classes case is predominant.

In the 4 solution classes, there is no (general) formula giving the other two fundamental solutions (i.e., those different from $(\pm 1, 2(3n-1))$ and $(\pm 1, 2(3n-2))$).

Examples. For the $(-)$ equation: $n = -6$, $(\pm 1, 40)$ and also $(\pm 7, 92)$, or, $n = -8$, $(\pm 1, 52)$ and also $(\pm 8, 137)$, $n = -7$, $(\pm 1, 46)$ and also $(\pm 4, 71)$.

For the $(+)$ equation: $n = -5$, $(\pm 1, 32)$ and also $(\pm 6, 67)$, or, $n = -10$, $(\pm 1, 62)$ and also $(\pm 2, 71)$, $n = 3$, $(\pm 1, 16)$ and also $(\pm 2, 19)$.

5 Appendix 2

In this section we mention two well-known reductions for the quadratic Diophantine equations displayed in Proposition 18. One deals with those in Proposition 19, in a similar manner.

Suppose $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(U) = 1$. Then (see Proposition 18), the equations ($\pm 3_{U-E_{11}}$) are

$$bx^2 - (a-d-1)xy - cy^2 + bx + (2d-1\pm 1)y = 0.$$

We first focus on $(+3)$.

Recall that any quadratic Diophantine equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (*)$$

can be reduced to a Pell equation

$$X^2 - \mathbf{D}Y^2 + 4A\Delta = 0,$$

using the transformations $X = \mathbf{D}y + 2AE - BD$, $Y = 2Ax + By + D$, where $\mathbf{D} = B^2 - 4AC$ and $\boldsymbol{\Delta} = 4ACF + BDE - AE^2 - CD^2 - FB^2$.

Coming back to our notations, $A = D = b$, $B = -(a-d-1)$, $C = -c$, $E = 2d$, $F = 0$, $\mathbf{D} = (a-d-1)^2 + 4bc = (a-d-1)^2 + 4(ad-1) = (a+d)^2 - 2a + 2d - 3$ and

$\boldsymbol{\Delta} = -2(a-d-1)bd - 4bd^2 + b^2c = b[-2(a-d-1)d - 4d^2 + bc] =$
$= b(-2ad - 2d^2 + 2d + ad - 1) = -b(ad + 2d^2 - 2d + 1)$ and so
$X = [(a+d)^2 - 2a + 2d - 3]y + 3bd + b(a-1)$, $Y = 2bx - (a-d-1)y + b$

Hence

Proposition 29. *Suppose* $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $\det(U) = 1$. *The equation* ($+3_{U-E_{11}}$) *is solvable for* x, y *over the integers iff the Pell equation*

$$X^2 - [(a+d)^2 - 2a + 2d - 3]Y^2 = b^2(ad + 2d^2 - 2d + 1)$$

has integer solutions and the equations $X = [(a+d)^2 - 2a + 2d - 3]y + 3bd + b(a-1)$, $Y = 2bx - (a-d-1)y + b$ *are solvable for* x, y *over the integers.*

Another reduction (and this is how [14] is constructed, with the same $\mathbf{D}$) is by using transformations $\mathbf{D}x = X + \alpha$, $\mathbf{D}y = Y + \beta$ with (the general notations from (*) above) $\alpha = 2CD - BE$, $\beta = 2AE - BD$. This way, another special quadratic equation is obtained (with the same degree two coefficients), namely

$$AX^2 + BXY + CY^2 = N,$$

which has several methods of solving over the integers, well-known in Number Theory. Here $-N = A\alpha^2 + B\alpha\beta + C\beta^2 + \mathbf{D}(D\alpha + E\beta + F\mathbf{D})$.

Coming back again to our notations (for ($+3_{U-E_{11}}$)), $\alpha = 2(1-d-d^2)$ and $\beta = b(a+3d-1)$ and so N can be accordingly computed (we skip the complicated formula). Hence

Proposition 30. *Suppose* $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $\det(U) = 1$. *The equation* $(+3_{U-E_{11}})$ *is solvable for* x, y *over the integers iff the equation*

$$bX^2 - (a-d-1)XY - cY^2 = N$$

has integer solutions and the equations $[(a+d)^2 - 2a + 2d - 3]x = X + 2(1 - d - d^2)$, $[(a+d)^2 - 2a + 2d - 3]y = Y + b(a + 3d - 1)$ *are solvable for* x, y *over the integers.*

For $(-3_{U-E_{11}})$ we just replace above E by $2(d-1)$ instead of $2d$.
Similar computation covers the $\det(U) = -1$ case.

Examples ([14] was used) with $\mathbf{D} > 0$ (i.e., the hyperbolic case) and $\det(U) = 1$. Since for a large amount of units U, and especially for those with $c \in \{\pm b\}$, the quadratic Diophantine equations $(\pm 3_{U-E_{11}})$ have infinitely many solutions, given by *two recurrence relations*, it is another hard task to check if at least one, satisfies the corresponding equation $(\pm 2_{U-E_{11}})$. This partly explains, the hard work necessary for proving special cases of the conjecture stated in the Subsection 3.2.

1) If $U = \begin{bmatrix} 3 & 10 \\ 5 & 17 \end{bmatrix}$ $(\det(U) = 1)$ then for $U - E_{11}$ we have:

(+3) the coefficients $(10, 15, -5, 10, 34, 0)$, $\mathbf{D} = 425$, $\alpha = -610$, $\beta = 530$, $425x = X - 610$, $425y = Y + 530$ and we solve $10X^2 + 15XY - 5Y^2 = -2533000$ (or the primitive equation $2X^2 + 3XY - Y^2 = -506600$).

The solution recurrences: $x_{n+1} = -39129x_n - 69680y_n + 30732$, $y_{n+1} = -139360x_n - 248169y_n + 109460$, respectively, $x_n = -248169x_{n+1} + 69680y_{n+1} - 443092$, $y_n = 139360x_{n+1} - 39129y_{n+1} + 248820$.

For (-3): $(10, 15, -5, 10, 32, 0)$, $\alpha = -580$, $\beta = 490$, $425x = X - 580$, $425y = Y + 490$ and we solve $10X^2 + 15XY - 5Y^2 = -2099500$ (or the primitive equation $2X^2 + 3XY - Y^2 = -419900$).

The solution recurrences: $x_{n+1} = -39129x_n - 69680y_n + 26936$, $y_{n+1} = -139360x_n - 248169y_n + 95940$ respectively $x_n = -248169x_{n+1} + 69680y_{n+1} - 419016$, $y_n = 139360x_{n+1} - 39129y_{n+1} + 235300$.

However, (± 2): $5(-3x + y + 2z) = 33 \pm 1$ has no integer solutions, so $U - E_{11}$ is not (nontrivially) clean.

2) If $U = \begin{bmatrix} 3 & -7 \\ -7 & 16 \end{bmatrix}$ $(\det(U) = -1)$ then for $U - E_{11}$ we have:

(+3) $(-7, 14, 7, -7, 34, 0)$, $\mathbf{D} = 392$, $\alpha = -574$, $\beta = -378$, $392x = X - 574$, $392y = Y - 378$ and we solve $-7X^2 + 14XY + 7Y^2 = 1731464$ (or the primitive equation $-X^2 + 2XY + Y^2 = 247352$).

The solution recurrences: $x_{n+1} = 5741x_n - 13860y_n - 4960$, $y_{n+1} = -13860x_n + 33461y_n + 11970$, respectively, $x_n = 33461x_{n+1} + 13860y_{n+1} + 62360$, $y_n = 13860x_{n+1} + 5741y_{n+1} + 25830$.

$(-7, 14, 7, -7, 32, 0)$, $\alpha = -546$, $\beta = -350$, $392x = X - 546$, $392y = Y - 350$ and we solve $-7X^2 + 14XY + 7Y^2 = 1446088$ (or the primitive equation $-X^2 + 2XY + Y^2 = 206584$).

The solution recurrences: $x_{n+1} = 5741x_n - 13860y_n - 4380$, $y_{n+1} = -13860x_n + 33461y_n + 10570$, respectively, $x_n = 33461x_{n+1} + 13860y_{n+1} + 58980$, $y_n = 13860x_{n+1} + 5741y_{n+1} + 24430$.

However (± 2): $7(-2x - y - z) = 33 \pm 1$ has no integer solutions, so $U - E_{11}$ is not (nontrivially) clean.

There are two cases, when we can easily decide the noncleanness of $U - E_{11}$.

In the elliptic case, i.e., $\mathbf{D} < 0$, and whenever $\mathbf{D}$ is a square (incl. $\mathbf{D} = 0$). In both cases, the equations $(\pm 3_{U-E_{11}})$ have finitely many solutions, so we can check for each of these $(\pm 2_{U-E_{11}})$ and decide. We mention that (at least statistically) these are rare.

From a total of 109335 nonclean matrices $U - E_{11}$ (with x, y, z bounded by 127), only 1604 (i.e., 1.47%) have $\mathbf{D} < 0$, and 3550 (i.e., 3.25%) have a square $\mathbf{D}$ (incl. 88, i.e., 0.88% with $\mathbf{D} = 0$). Therefore, in order to decide the cleanness, hard work is necessary for some 95.2% of the candidates given by computer (including the two examples above). By hard work, we mean (as already done in Theorem 26, Appendix 1), we start with the linear Diophantine equations (± 2) and replace their solutions into the corresponding (± 3).

Examples. 5) If $U = \begin{bmatrix} 17 & -24 \\ 12 & -17 \end{bmatrix}$ then for $U - E_{11}$ we have:

(+3) $(-24, -33, -12, -24, -32, 0)$ $\mathbf{D} = -63$, $\alpha = -480$, $\beta = 744$, $-63x = X - 480$, $-63y = Y + 744$ and we solve $-24X^2 - 33XY - 12Y^2 = -387072$ (or the primitive equation $-8X^2 - 11XY - 4Y^2 = -129024$). Three solutions: $(0, 0)$, $(-1, 0)$ and $(16, 24)$. None verifies (+2).

(-3) $(-24, -33, -12, -24, -34, 0)$ $\mathbf{D} = -63$, $\alpha = -546$, $\beta = 840$, $-63x = X - 546$, $-63y = Y + 840$ and we solve $-24X^2 - 33XY - 12Y^2 = -486864$ (or the primitive equation $-8X^2 - 11XY - 4Y^2 = -162288$). Five solutions: $(0, 0)$, $(-1, 0)$, $(18, -27)$, $(14, -18)$, $(10, -12)$. None verifies (-2).

6) If $U = \begin{bmatrix} -7 & -9 \\ 3 & 4 \end{bmatrix}$ then for $U - E_{11}$ we have:

(+3) $(-9, 12, -3, -9, 10, 0)$ $\mathbf{D} = 36$, $\alpha = -66$, $\beta = -72$, $36x = X - 66$, $36y = Y - 72$ and we solve $-9X^2 - 12XY - 3Y^2 = 2268$, i.e. $(-X + Y)(-3X + Y) = 2268$. Only two solutions: $(0, 0)$, $(-1, 0)$ solutions. None satisfies $(+2_{U-E_{11}})$: $3(-4x + y - 3z - 3) = 1$.

(-3) $(-9,12,-3,-9,8,0)-9x^2+12xy-3y^2-9x+8y=0$ $\mathbf{D}=36$, $\alpha=-42$, $\beta=-36$, $36x=X-42$, $36y=Y-36$ and we solve $-9X^2+12XY-3Y^2=-1620$, i.e. $(-X+Y)(-3X+Y)=540$. Only two solutions: $(0,0)$, $(-1,0)$ solutions. None satisfies $(-2_{U-E_{11}})$: $3(-4x+y-3z-3)=-1$.

7) If $U=\begin{bmatrix} 2 & -3 \\ 3 & -5 \end{bmatrix}$ then for $U-E_{11}$ we have:

(+3) $(-3,-6,-3,-3,-8,0)$ $\mathbf{D}=0$ and we have four families of solutions: (i) $x=60w^2+44w+7$, $y=-60w^2-54w-12$, (ii) $x=60w^2-4w-1$, $y=-60w^2-6w$, (iii) $x=60w^2-64w+16$, $y=-60w^2+54w-12$ and (iv) $x=60w^2-16w$, $y=-60w^2+6w$, with arbitrary integer w.

Analogously for (-3): also four families of solutions.

However, (± 2) $3(2x+y-z+3)=\pm 1$, have no integer solutions.

Remark. More can be proved for *strongly clean* matrices in all cases, since $E_{11}U=UE_{11}$ forces the unit U to be diagonal with entries ± 1. In order not to further lengthen this paper, we leave this for the reader.

In closing we introduce two subclasses of negative clean rings.

A ring is called *sum clean* if sums of clean elements are also clean (equivalently, $\mathrm{cn}(R)+\mathrm{cn}(R)\subseteq \mathrm{cn}(R)$). Obviously, clean rings are also sum clean. In a sum clean ring, sums of idempotents and sums of units are clean.

Using the fact that $-e=(1-e)+(-1)\in \mathrm{cn}(R)$ for every idempotent e, we can deduce that *sum clean rings are negative clean*. Indeed if $a=e+u$ then $-a=-e-u\in \mathrm{cn}(R)+\mathrm{cn}(R)\subseteq \mathrm{cn}(R)$. *In a sum clean ring,* $\mathrm{cn}_1(R)$ *is an additive subgroup of* $(R,+)$.

A ring is called *product clean* if products of clean elements are also clean (equivalently, $\mathrm{cn}(R)\cdot \mathrm{cn}(R)\subseteq \mathrm{cn}(R)$). Obviously, clean rings are also product clean. Since $-1\in U(R)\subseteq \mathrm{cn}(R)$, *product clean rings are negative clean*.

Since unit-regular elements can be represented as products eu with $e^2=e$ and $u\in U(R)$, *in a product clean ring, unit-regular elements are clean*.

As already noticed for negative clean rings, *sum (or product) clean rings need not be exchange*,

Sum clean and product clean rings will be addressed elsewhere.

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