THE RINGS ALL WHOSE NIL-CLEAN ELEMENTS ARE UNIQUELY NIL-CLEAN ARE PRECISELY THE ABELIAN RINGS

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In [6] rings with the property that *every regular element is strongly regular* are studied. Analogously, one could study rings whose *clean elements are strongly clean*, or rings whose *nil-clean elements are strongly nil-clean*, or rings whose *fine elements are strongly fine*. Or else, a study may be developed replacing "strongly" with "uniquely".

Since in a ring, *clean elements are strongly clean iff units and idempotents commute*, and, *nil-clean elements are strongly nil-clean iff idempotents and nilpotents commute*, we can immediately discard two possibilities: rings whose idempotents commute with nilpotents or idempotents commute with units turn out (see [4], ex.12.7) to be precisely the so called *Abelian rings* (i.e., rings with only central idempotents).

Fine elements are strongly fine in a ring iff units commute with nilpotents. Such rings, called *uni rings*, were studied in [1].

Therefore the "strongly" version is settled and in this note we address the "uniquely" version.

The rings all whose clean elements are uniquely clean (called CUC rings) were studied in [2]. As already mentioned in [2], *all fine elements in a ring R are uniquely fine iff the ring R is reduced.*

In this note, we characterize the rings all whose nil-clean elements are uniquely nil-clean (called NUN rings). This class of rings also coincides with the Abelian rings.

All rings we consider are associative and unital. We denote $\overline{e} = 1 - e$ the *complementary* idempotent of an idempotent e.

We first generalize a bit, Lemma 1.3 from [5]

Proposition 1. *If the idempotents of a ring are uniquely nil-clean then the ring is Abelian.*

Proof. Let $e \in R$ be an idempotent and let r be any element of R. Notice that the idempotent $e + er\overline{e}$ can be written as nil-clean $e + (er\overline{e})$ or as $(e + er\overline{e}) + 0$. Since R is uniquely nil clean, this shows that $er\overline{e} = 0$ or $er = ere$. Using the idempotent $e + \overline{e}re$, it can likewise be shown that $\overline{e}re = 0$ or $re = ere$, and so e is central. □

Corollary 2. *NUN rings are Abelian.*

The converse is also true, so NUN ≡ Abelian.

Proposition 3. *Abelian rings are NUN.*

Proof. If a ring R is Abelian, all nil-clean elements are strongly nil-clean. By Corollary 3.8 [3], if an element of a ring is strongly nil clean, then it has precisely one strongly nil clean decomposition. Therefore, (strongly) nil-clean elements are also uniquely (strongly) nil-clean. Hence R is NUN. □

References

[1] G. Călugăreanu *Rings whose units commute with nilpotent elements*. Mathematica **60** (83), (2) 2018), 119-126.

[2] G. Călugăreanu, Y. Zhou *Rings all whose clean elements are uniquely clean*. submitted

[3] A. J. Diesl *Nil clean rings*. J. of Algebra **383** (2013) 197-211.

[4] T. Y. Lam *A first course in noncommutative rings*. Graduate Texts in Math. **131**, Springer 2001.

[5] S. Sahebia, M. Jahandar *A note on uniquely (nil) clean ring*. Journal of Linear and Topological Algebra **1** (2) (2013), 67- 69.

[6] C. Wu, Zhao *RS rings and their applications*. Publ. Math. Debrecen **3** (2020), 1-14.