AN IDEMPOTENT NOT CONJUGATE WITH ITS COMPLEMENTARY IDEMPOTENT

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Abstract. In $\mathbb{M}_2(\mathbb{Z}[i\sqrt{5}])$, we show that the idempotent $\begin{bmatrix} 3 & \alpha \\ -\overline{\alpha} & -2 \end{bmatrix}$ with $\alpha = 1 + i\sqrt{5}$ is not similar to its complementary idempotent.

1. Introduction

Clearly, searching for examples of idempotents as in the title, makes sense only for *nontrivial* idempotents, that is $e^2 = e \notin \{0, 1\}$.

We also mention that in any nonzero ring, every idempotent is different from its complementary idempotent.

The example we provide is a 2×2 matrix over a domain.

Recall that a domain D is *GCD* if greatest common divisors exist for every pair of elements of D. Over many types of rings, the nontrivial idempotent 2×2 matrices are all similar to E_{11} (here E_{ij} denotes the matrix units, that is, matrices with all entries zero excepting the (i, j)-entry which is 1). The binary relation of similarity being transitive and symmetric, it follows that in such matrix rings actually all nontrivial idempotents are similar, and so are in particular, every nontrivial idempotent and its complementary idempotent.

Therefore examples as in the title should be possibly found in matrix rings over domains that are not GCD.

This will be our case for $\mathbb{Z}[i\sqrt{5}]$.

In Section 2 we present matrix rings which should be avoided when searching for an example and in Section 3 we provide the example of nontrivial idempotent which is not similar to its complementary idempotent, namely $E = \begin{bmatrix} 3 & \alpha \\ -\overline{\alpha} & -2 \end{bmatrix}$ where $\alpha = 1 + i\sqrt{5}$ (here $\overline{\alpha}$ denotes the (complex) conjugate of α).

2. 2×2 matrices

The following result is known. For readers convenience, we supply a proof.

Proposition 1. *Any non-trivial* 2×2 *idempotent matrix over a GCD domain is similar to* E_{11}.

Proof. Let D be a GCD domain and let $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} \in \mathbb{M}_2(D)$ be a nontrivial idempotent, i.e. $bc = a(1 - a)$.

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First, for $a = 0$, at least one of b, c is zero, say $b = 0$ (the $c = 0$ case is analogous). Then $E = \begin{bmatrix} 0 & 0 \\ c & 1 \end{bmatrix}$ and for $U = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix}$ one checks that $E_{11} = U^{-1}EU$.

Next, assume $a \neq 0$ and let $x = \gcd(a, c)$. If $a = xy$ and $c = xx'$ it follows that $\gcd(y, x') = 1$. By cancellation with x we get $bx' = y(1 - a)$, and so, by our last hypothesis, y divides b, say $b = yy'$. We also have $x'y' = 1 - a$. Now take $P = \begin{bmatrix} x & y' \\ -x' & y \end{bmatrix}$. One can check that $\det(P) = 1$ and $PE = E_{11}P$. Hence E is similar to E_{11}. □

Corollary 2. *Over a GCD domain any two nontrivial* 2×2 *idempotent matrices are similar. In particular, any nontrivial idempotent is similar to its complementary idempotent.*

This result can be further generalized.

Following Steger [1], we say that a ring R is an *ID* ring if every idempotent matrix over R is similar to a diagonal one.

Examples of ID rings include: division rings, local rings, principal ideal domains, elementary divisor rings, unit-regular rings and serial rings.

A ring is called *connected* if it has only trivial idempotents. Then we obtain

Proposition 3. *Over any ID connected ring, any two nontrivial* 2×2 *idempotent matrices are similar.*

3. Example in $\mathbb{Z}[i\sqrt{5}]$

The commutative domain $\mathbb{Z}[i\sqrt{5}]$ is mostly known as an example of not UFD (unique factorization domain), due for example to the non associate decompositions

$$3 \cdot 2 = (1 + i\sqrt{5})(1 - i\sqrt{5}).$$

Recall, as a useful tool, the so called *norm* of elements in $\mathbb{Z}[i\sqrt{5}]$, a multiplicative function $N : \mathbb{Z}[i\sqrt{5}] \longrightarrow \mathbb{N}$. Thus if $a \mid b$ also $N(a) \mid N(b)$ and using the norm it also follows that the units of $\mathbb{Z}[i\sqrt{5}]$ are ± 1.

Also recall that $\mathbb{Z}[i\sqrt{5}]$ is not a GCD domain. For example, $\gcd(6, 2(1 + i\sqrt{5}))$ does not exist.

Lemma 4. *(i) Over* $\mathbb{Z}[i\sqrt{5}]$*, the equation* $yz = 2$ *has only integer solutions, i.e.,* $y, z \in \{\pm 1, \pm 2\}$.

(ii) Excepting $1 \cdot 6 = 6 \cdot 1$*, the only product decompositions of* 6 *in* $\mathbb{Z}[i\sqrt{5}]$ *are* $3 \cdot 2 = 2 \cdot 3 = (1 + i\sqrt{5})(1 - i\sqrt{5}) = (1 - i\sqrt{5})(1 + i\sqrt{5})$.

Proof. (i) Using the norm, $4 = N(2) = N(yz) = N(y)N(z) = (a^2 + 5b^2)(c^2 + 5d^2)$ for integers a, b, c, d is possible only if $b = d = 0$.

(ii) Any decomposition of $6 = xy$ gives a decomposition of $N(6) = 36 = N(x)N(y)$ as product of norms. Assuming $N(x) \leq 6 \leq N(y)$, as $N(x)$ cannot be 2 and 3, it follows that $N(x)$ is 1 or 4 or 6, and the conclusion follows as the equations $N(x) = 1$, $N(x) = 4$ and $N(x) = 6$ can be easily solved, and $y = \dfrac{6}{x}$. □

Taking into account Proposition 1, we may ask whether the idempotent matrix $E = \begin{bmatrix} 3 & 1 + i\sqrt{5} \\ -1 + i\sqrt{5} & -2 \end{bmatrix}$ over $\mathbb{Z}[i\sqrt{5}]$, is similar E_{11} and/or is similar to its complementary idempotent. We answer both questions in the negative below, so that this is the desired example.

To simplify the writing in the sequel we denote $\alpha = 1 + i\sqrt{5}$ so that $E = \begin{bmatrix} 3 & \alpha \\ -\overline{\alpha} & -2 \end{bmatrix}$.

Theorem 5. *Over $\mathbb{Z}[i\sqrt{5}]$, E is not similar to $I_2 - E$.*

Proof. For $U = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ with $xw - yz = \pm 1$ and $EU = UE_{11}$, we reduce (two out of four equations are dependent) to only $x + w = 0$ and $5x + \alpha z = \overline{\alpha} y$. Therefore it remains (we eliminate w) to show that the system

$$5x + \alpha z = \overline{\alpha} y, x^2 + yz = \pm 1$$

has no solutions in $\mathbb{Z}[i\sqrt{5}]$.

We can assume $xw - yz = -1$ (otherwise, we just replace, say, x, z by $-x, -z$). Multiplying the linear equation by α we get $5\alpha x + \alpha^2 z = 6y$. Replacement in $6x^2 + 6yz = 6$ leads to $(2x + \alpha z)(3x + \alpha z) = 6$ (the discriminant of the quadratic equation is a square, namely α^2). So an equivalent system is now

$$5x + \alpha z = \overline{\alpha} y, (2x + \alpha z)(3x + \alpha z) = 6.$$

Since (see Lemma 4, (ii)) the only decompositions of 6 are $2 \cdot 3 = 3 \cdot 2 = \alpha\overline{\alpha} = \overline{\alpha}\alpha$ (excepting $1 \cdot 6 = 6 \cdot 1$ which give units, not our case), we have to solve four linear systems.

The first one is

$$2x + \alpha z = 2, 3x + \alpha z = 3, 5x + \alpha z = \overline{\alpha} y.$$

From the first two equations we get $x = 1$, $z = 0$ which implies $\overline{\alpha} y = 5$, that is $\overline{\alpha}$ would divide 5. As $\overline{\alpha} \mid 6$ and $\overline{\alpha}$ is not a unit, we have a contradiction.

The second is

$$2x + \alpha z = 3, 3x + \alpha z = 2, 5x + \alpha z = \overline{\alpha} y.$$

From the first two equations we get $x = -1$ which implies $\alpha z = 5$. As $\alpha \mid 6$ we obtain a contradiction as in the previous case.

The third one is

$$2x + \alpha z = \alpha, 3x + \alpha z = \overline{\alpha}, 5x + \alpha z = \overline{\alpha} y.$$

From the first two equations we get $x = \overline{\alpha} - \alpha = -2i\sqrt{5}$ which implies $\alpha(z - 1) = 4i\sqrt{5}$, a contradiction as α does not divide $4i\sqrt{5}$ (this can be seen using the norms: if $u \mid v$ in $\mathbb{Z}[i\sqrt{5}]$ then $N(u) \mid N(v)$ in $\mathbb{Z}$).

Finally the fourth is

$$2x + \alpha z = \overline{\alpha}, 3x + \alpha z = \alpha, 5x + \alpha z = \overline{\alpha} y.$$

From the first two equations we get $x = \alpha - \overline{\alpha} = 2i\sqrt{5}$ and we continue as in the previous case. This completes our proof. □

Corollary 6. *Over $\mathbb{Z}[i\sqrt{5}]$, E is not similar to E_{11}.*

We provide two proofs for this consequence.

Proof. It is easy to see that for any (nontrivial) idempotent E, if $EU = UE_{11}$ then $(I_2 - E)U = UE_{22}$. Since $E_{22} = (E_{12}+E_{21})E_{11}(E_{12}+E_{21})$ it follows by transitivity that if E is similar to E_{11}, so is also its complementary $I_2 - E$. Now, would our idempotent E be similar to E_{11}, its complementary should also be similar to E_{11} and so by transitivity and symmetry we contradict the previous theorem. $\square$

Proof. A direct proof, not consequence of the previous theorem.

For $U = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ with $xw - yz = 1$ and $EU = UE_{11}$, we reduce (two out of four equations are dependent) to only $2x = -(1 + i\sqrt{5})z$ and $2w = (-1 + i\sqrt{5})y$. By multiplication, $4xw = 6yz$ and so from $4xw - 4yz = 4$ we get $yz = 2$, $xw = 3$.

If $z \in \{\pm 1\}$ we get $2 \mid 1 + i\sqrt{5}$, a contradiction. If $z \in \{\pm 2\}$ then $y \in \{\pm 1\}$ and we get $2 \mid -1 + i\sqrt{5}$, again a contradiction. Hence the three equations system has no solutions in $\mathbb{Z}[i\sqrt{5}]$. $\square$

References

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