Every two isomorphic idempotents of a ring are equal iff the ring is Abelian.

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Abstract

Using some information from [5], the result follows easily.

1 Isomorphic idempotents

Definition. Two idempotents e, f are *isomorphic* (written $e \cong f$) if $eR \cong fR$ as right R-modules, or equivalently, $Re \cong Rf$ as left R-modules. It is well-known (see [6], **21.20**) that *two idempotents e, f of a ring R are isomorphic iff there exist $a, b \in R$ such that $e = ab$ and $f = ba$)*.

Then it follows that

Lemma 1 *Conjugate idempotents are isomorphic.*

Proof. Indeed, if $f = (u^{-1}e)u$ then $e = u(u^{-1}e)$ and we use the above mentioned characterization. ■

Next we recall **Ex.22.2** with solution (see [5])

Lemma 2 *Two central idempotents e, f in a ring are isomorphic iff if these are equal.*

Proof. As equal idempotents are obviously isomorphic, suppose $e, f \in Z(R)$ (the center of R). Then $e = ab$, $f = ba$ and so $f = f^2 = b(ab)a = bea = e(ba) = ef$. Similarly $e = fe$. ■

Continue with **Ex.22.3A** with some additions [(vi) is also **Ex.12.7**].

Lemma 3 *The following conditions are equivalent:*

(i) $e \in Z(R)$,

(ii) $eR = Re$,

(iii) e commutes with all the idempotents,

(iv) e commutes with all the idempotents of R that are isomorphic to e,

(v) e commutes with all the units,

(vi) e commutes with all the nilpotents.

Proof. The additional (iii) is obvious. As for additional (v) and (vi) just notice that

$e \in Z(R) \Rightarrow e$ commutes with all units $\overset{unipotents}{\Rightarrow}$ e commutes with all nilpotents $\overset{\textbf{Ex.12.7}}{\Rightarrow}$ $e \in Z(R)$. ■

We are now ready to prove the statement in the title.

Theorem 4 *In a ring, the following conditions are equivalent.*
(a) any two isomorphic idempotents are equal,
(b) any two conjugate idempotents are equal,
(c) the ring is Abelian.

Proof. Since conjugate idempotents are isomorphic, (a) $\Longrightarrow$ (b).

If any two conjugate idempotents are equal, for every idempotent e and every $u \in U(R)$ we have $e = u^{-1}eu$. It follows that idempotents commute with units and so (b) $\Longrightarrow$ (c) follows from (v), the previous lemma.

If the ring is Abelian, (c) $\Longrightarrow$ (a) follows from Lemma 2. ■

1.0.1 Application

Proposition 5 *Let R be a regular ring. The following conditions are equivalent.*
(i) R is strongly regular;
(ii) R is Abelian (regular);
(iii) Any two isomorphic idempotents of R are equal.

Moreover, *in unit-regular rings two idempotents are conjugate iff these are isomorphic.*

And more detailed: in *a regular ring, every two isomorphic idempotents are equal iff the ring is unit-regular.*

Hence, to the above proposition we can add
(iv) R is unit-regular, and also
(vi) R is right (or left) duo (see next section).
[This is **Ex. 22.4B**].

2 Refinement for conjugate idempotents

We can add two refinements:

Definitions. Two idempotents e, f are *strongly right conjugated* if $eR = fR$.

Lemma 6 *The following conditions are equivalent.*
(1) $eR = fR$,
(2) $ef = f$, $fe = e$,
(3) $f = e + er\overline{e}$ for some $r \in R$,
(4) $f = eu$ for some $u \in U(R)$,
(5) $R\overline{e} = R\overline{f}$.

Remarks. 1) Actually, $ef = f$, $fe = e$ implies $f = e + ef\overline{e}$.

2) The above conditions imply that e, f are conjugated: for $u = 1 + er\overline{e}$ it follows $f = u^{-1}eu$. However, examples show that the converse fails (see **Ex. 21.4**, [5]).

Symmetrically, two idempotents e, f are *strongly left conjugated*.

Such pairs of idempotents are (indeed) conjugated: above $u = 1 + er\overline{e}$ so $u^{-1}e = (1 - er\overline{e})e = e$. Finally, $u^{-1}eu = eu = f$.

Therefore

$$\begin{array}{ccccccc} & & strong.right\ conjugation & & & & \\ & \nearrow & & \searrow & & & \\ equality & & & & conjugation & \rightarrow & isomorphy \\ & \searrow & & \nearrow & & & \\ & & strong.left\ conjugation & & & & \end{array}$$

According to the types of idempotents described above, we introduce the following

Definitions. A ring is called *CIE* if every two **c**onjugate **i**dempotents are **e**qual.

Moreover, a ring is *CIRSC* (resp. *CILSC*) if every **c**onjugate **i**dempotents are **r**ight (resp. left) **s**trongly **c**onjugate, and, is *RSCIE* (resp. *LSCIE*) if every right (resp. left) strongly conjugate idempotents are equal.

Clearly

Proposition 7 *A ring is CIE iff it is CIRSC and RSCIE iff it is CILSC and LSCIE.*

Now recall some other well-known (actually equivalent) definitions.

An element $a \in R$ is *right (left) subcommutative* if $Ra \subseteq aR$ ($aR \subseteq Ra$) and subcommutative if it is both left and right subcommutative. A subset of R is (left) (right) subcommutative if so is each of its elements.

The concept of one–sided subcommutativity frequently occurs in the ring theory literature under different names. For example, Birkenmeier [1, p. 569] defines an idempotent $e \in E$ to be *left (right) semicentral* if $Re = eRe$ ($eR = eRe$). It is readily seen that left semicentral is equivalent to our right subcommutative.

Reid [7, Section 3] gives an example of a non–commutative left subcommutative endomorphism ring. However, subcommutative idempotents are actually central.

It is readily checked that *an idempotent* e is left subcommutative if and only if eR has a unique complement. More

Proposition 8 *Let* $e \in Id(R)$. *The following are equivalent:*

(1) e *is left subcommutative.*

(2) $eR\overline{e} = 0$.

(3) $\overline{e}$ *is right subcommutative.*

(4) eR *has a unique complement.*

A related commutativity condition on rings is that one–sided ideals are two–sided.

Definition 9 *A ring is called right (left) duo if every right (left) ideal is two-sided.*

One shows that

Proposition 10 *A ring is right (left) duo if and only if it is right (left) subcommutative.*

As *nowadays* the term right (or left) duo is merely used, this is what we use in the sequel.

Using Lemma 6, (3) we can show

Proposition 11 *A ring is RSCIE (or LSCIE) iff the idempotents are left (resp. right) subcommutative.*

Proof. Indeed, (say) for the *right* part, $f = e + er\overline{e} = e$ iff $eR\overline{e} = 0$ and we use Proposition 10. ■

Also recall that *every right (resp. left) duo ring is Abelian* (**Ex. 22.4A**, [5]).

This does **not** folow from our termnology above:

left duo $\Leftrightarrow$ left subcommutative $\Rightarrow$ idempotents are left subcommutative $\Leftrightarrow$ RSCIE,

which generally does not imply CIE (equivalent to Abelian).

Further, using Lemma 6 (4), we get

Proposition 12 *A ring is CIRSC (or CILSC) iff for every two conjugated idempotents e, f there is a unit u such that $f = eu$ (resp. $f = ue$).*

References

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