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Rings with fine idempotents

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An idempotent in a ring is called fine (see G. Călugăreanu and T. Y. Lam, Fine rings: A new class of simple rings, *J. Algebra Appl.* **15**(9) (2016) 18) if it is a sum of a nilpotent and a unit. A ring is called an idempotent-fine ring (briefly, an IF ring) if all its nonzero idempotents are fine. In this paper, the properties of IF rings are studied. A notable result is proved: The diagonal idempotents $E_{11} + \cdots + E_{ii}$ $(i = 1, \ldots, n)$ are fine in the matrix ring $\mathcal{M}_n(R)$ for any unital ring R and any positive integer n. This yields many classes of rings over which matrix rings are IF.

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1. Introduction

Rings in which *every unit is nil-clean* (that is, every unit is a sum of an idempotent and a nilpotent) are studied in [8] under the name of *UNC* rings, a generalization of the UU rings (see [2, 6]). In [4], *fine* elements in a ring were introduced as sums of a nilpotent and a unit. Having in mind these three important subsets in any ring, the units, the idempotents and the nilpotents, there are two other possible combinations: *Every nilpotent is clean* (that is, every nilpotent is a sum of an idempotent and a unit), which holds in any ring, and *every idempotent is fine*.

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Since 0 is not fine, it must be excepted. On the contrary, 1 is always fine: $1 = t + (1 - t)$ for any nilpotent t. The rings all whose *nonzero* idempotents are fine will be called *IF* rings. As noticed above, in order to check whether a ring is IF, it suffices to deal with the nontrivial idempotents. From the definition, *reduced rings with nontrivial idempotents are not IF* and *any ring with only trivial idempotents is IF*. In particular local rings and domains are IF. *A commutative ring is IF iff it has only the trivial idempotents.* So in the sequel, for the study of IF rings, we may discard the commutative rings.

A nonzero ring R is called *idempotent-simple* if for any nonzero idempotent e of R, $ReR = R$.

In this paper, Sec. 2 deals with the general properties of the IF rings and Sec. 3 studies the IF property for matrix rings. The main results are: The class of IF rings is a subclass of idempotent-simple rings and it contains the class of Artinian idempotent-simple rings properly; over any unital ring R, the diagonal idempotents $E_{11} + \cdots + E_{ii}$ $(i = 1, \ldots, n)$ are fine in the matrix ring $\mathcal{M}_n(R)$ for any positive integer n, with many consequences giving classes of rings over which matrix rings are IF; a special attention is given to the $n = 2$ case: Over GCD (every pair of elements has a greatest common divisor) domains, 2×2 matrix rings are IF.

Throughout, R is an associative unital ring. We denote by $J(R)$, $U(R)$, $Z(R)$ and $\text{nil}(R)$ the Jacobson radical, the unit group, the center and the set of nilpotents of R, respectively. We write $\mathcal{M}_n(R)$ for the ring of all $n \times n$ matrices over R whose identity is denoted by I_n and $\mathcal{T}_n(R)$ for the ring of all upper triangular $n \times n$ matrices over R. Let $E_{ij} \in \mathcal{M}_n(R)$ be the standard matrix unit, i.e. E_{ij} has a 1 in the (i, j) position and zeros in all other positions. As in [4], $\Phi(R)$ denotes the set of all fine elements of R.

2. IF Rings

First, we recall from [4].

Lemma 1. (1) *For any ring R, $\Phi(R) \cap Z(R) = U(Z(R))$.*
(2) *If $a \in \Phi(R)$, then $RaR = R$.*

As a consequence of Lemma 1(2), *in an IF ring, all idempotents are full.* Next, a simple but important result.

Proposition 2. *Every IF ring is indecomposable*; *but an indecomposable ring need not be an IF ring.*

Proof. This is because, in an IF ring, 1 is the only central idempotent that is fine (see Lemma 1(1)). The indecomposable ring $\mathcal{T}_n(F)$ (where $n \geq 2$ and F is a field) is not an IF ring (see Corollary 4(5)). □

Thus, any direct product of two or more IF rings is not an IF ring, and the center of an IF ring is an IF ring. In another direction, we can improve an observation (about commutative rings) made in Sec. 1, as follows: *An Abelian ring* (i.e. *idempotents are central*) *is IF iff it has only the trivial idempotents.*

Factor rings of IF rings may not be IF: $\mathbb{Z}$ is IF, but $\mathbb{Z}_6$ is not (commutative with nontrivial idempotents). However, we do have the following proposition.

Proposition 3. *Let R be a ring, and I an ideal of R.*

(1) *Suppose that idempotents lift modulo I in R. If R is an IF ring, then R/I is an IF ring.*
(2) *Suppose that I is nil. Then R is an IF ring iff R/I is an IF ring.*

Proof. (1) Let $\bar{e}$ be a nonzero idempotent in R/I. As idempotents lift modulo I, we can assume that e is a (nonzero) idempotent of R. So, $e = b + u$, where b is nilpotent and u is a unit. Therefore, $\bar{e} = \bar{b} + \bar{u}$ is fine in R/I.

(2) The necessity is by (1). For the sufficiency, let e be a nonzero idempotent of R. So $\bar{e} \in R/I$ is a nonzero idempotent. Write $\bar{e} = \bar{b} + \bar{u}$ where $\bar{b} \in \text{nil}(R/I)$ and $\bar{u}$ is a unit in R/I. Thus, $e = b + (u + j)$ for some $j \in I$. Since I is nil, $b \in \text{nil}(R)$ and u is a unit of R it follows that $u + j$ is a unit of R. So e is fine. □

Corollary 4. *Let R be a ring.*

(1) *For a nontrivial bimodule M over R, the trivial extension $R \propto M$ is an IF ring iff R is an IF ring.*
(2) *For $n \geq 2$, $R[t]/(t^n)$ is an IF ring iff R is an IF ring.*
(3) *Let σ be a ring endomorphism of R with $\sigma(1) = 1$. Then the left skew power series ring $R[[t;\sigma]]$ is an IF ring iff R is an IF ring.*
(4) *Let R, S be nontrivial rings and M an (R, S)-bimodule. The formal triangular matrix ring $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ is not an IF ring.*
(5) *For $n \geq 2$, $\mathcal{T}_n(R)$ is not an IF ring.*

Proof. (1) Let $S = R \propto M$ and $I = 0 \propto M$. Then I is a nil ideal of S and $S/I \cong R$. So, the claim follows by Proposition 3(2).

(2) Let $S = R[t]/(t^n)$ and $I = (t)/(t^n)$. Then I is a nil ideal of S and $S/I \cong R$. So, the claim follows from Proposition 3(2).
(3) Let $S = R[[t;\sigma]]$ and $I = St$. Then $S/I \cong R$ and idempotents lift modulo I in S. So, the sufficiency follows from Proposition 3(1). For the necessity, let $e := e_0 + e_1 t + e_2 t^2 + \cdots$ be a nonzero idempotent in S. Then $e_0 \in R$ is a nonzero idempotent, so $e_0 = b + u$ is a sum of a nilpotent and a unit. Thus, as $St \subseteq J(S)$, $e = b + (u + e_1 t + e_2 t^2 + \cdots)$ is a sum of a nilpotent and a unit in S.
(4) Let $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ and $I = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$. Then I is a nil ideal of T and $T/I \cong R \times S$, which is not an IF ring. So T is not an IF ring by Proposition 3.
(5) Argue as in (4).

□

An element $a \in R$ is a unipotent if $a \in 1 + \text{nil}(R)$. The next lemma is obvious.

Lemma 5. *Let $a \in R$. Then a is a fine element iff $1 - a$ is the sum of a unipotent and a unit.*

Corollary 6. *A ring R is an IF ring iff every nonidentity idempotent is the sum of a unipotent and a unit.*

Remarks.(1) *In an IF ring, the identity need not be the sum of two units*; for example, consider $\mathbb{Z}_2$.

(2) *The IF property does not pass on to subrings.* For a division ring D and $n \geq 2$, $\mathcal{M}_n(D)$ is an IF ring (later) but $\mathcal{T}_n(D)$ is not an IF ring.

(3) *For an IF ring, idempotents do not need to lift modulo its Jacobson radical.* Let $R = \{\frac{m}{n} \in \mathbb{Q} : 2 \nmid n \text{ and } 3 \nmid n\}$. Then R is an IF ring (a domain indeed), but idempotents do not lift modulo $J(R)$ (see [1, p. 312]).

(4) *The IF property of a ring R does not pass, in general, to $R/J(R)$.* Indeed, let R be the ring in the previous remark. Then R is an IF ring, but $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ is not an IF ring.

(5) *The complementary idempotent of a nonzero fine idempotent may not be fine.* Indeed, let $R = A \times \mathcal{M}_2(B)$, where A, B are nontrivial rings. Let $e = \left(1, \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right)$. Then $e = \left(0, \left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right]\right) + \left(1, \left[\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right]\right)$ is the sum of a nilpotent and a unit, so e is a nontrivial fine idempotent of R. But $1 - e = \left(0, \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right)$ is not fine, as $0 \in A$ is not fine.

(6) *Starting with a fine idempotent e, lots of other fine idempotents can be constructed.* Indeed, if $e^2 = e \in R$ and $f = 1 - e$, consider $e_0 := e + erf$ where $r \in R$. Then e_0 is an idempotent and $(1 + erf)e_0 = e_0 = e(1 + erf)$. Since $1 + erf$ is a unit, e and e_0 are conjugate. Thus, e is fine iff e_0 is fine. In other words, e being fine implies that $e + erf$ is fine for every $r \in R$.

Recall that a nonzero ring R is called *idempotent-simple* if for any nonzero idempotent e of R, $ReR = R$. Examples of idempotent-simple rings include simple rings and IF rings (in particular, fine rings). Idempotent-simple rings are indecomposable, but indecomposable rings need not be idempotent-simple: For example $\mathcal{T}_n(F)$ (where $n \geq 2$ and F is a field).

Proposition 7. *Every nonzero IF ring is idempotent-simple.*

Proof. Any nonzero idempotent e of R is fine, so $ReR = R$ by Lemma 1(2). Hence R is idempotent-simple. □

In [4], it was proved that simple Artinian rings are fine and fine rings are simple, both inclusions being strict. Something similar happens in our context.

Proposition 8. *Every left Artinian idempotent-simple ring R is IF.*

Proof. Since R is left Artinian, $\overline{R} := R/J(R) = T_1 \oplus \cdots \oplus T_k$ where each T_i is the matrix ring over a division ring. Let x be the identity of T_1. Since idempotents lift modulo $J(R)$, there exists $0 \neq e^2 = e \in R$ such that $\bar{e} = x$. Since R is idempotent-simple, $R = ReR$, and it follows that $\overline{R} = \overline{R}\bar{e}\overline{R} = T_1$, the matrix ring over a division ring D. Since idempotents lift modulo $J(R)$, any set of matrix units of $R/J(R)$ is lifted to a set of matrix units of R (for details see [9, Theorem 23.10]). Hence, R is the matrix ring over a local ring S with $S/J(S) \cong D$. So R is IF. □

However, a nonzero IF ring need not be idempotent-simple Artinian: Just take any domain which is not division ring. Fine rings are simple (see [4]). But we do not know whether every simple ring is IF.

Following Steger [11], we say that a ring R is an *ID* ring if every idempotent matrix over R is similar to a diagonal one. Thus, by a result of Song and Guo [10, Corollary 5], if every matrix over R is equivalent to a diagonal matrix, then R is an *ID* ring. Examples of *ID* rings include: Division rings, local rings, projective-free rings, principal ideal domains, elementary divisor rings, unit-regular rings and serial rings.

Proposition 9. *Let R be idempotent-simple and $n \geq 1$.*

(1) *For each $e^2 = e \in R$, eRe is idempotent-simple.*
(2) *If R is an ID ring, then $\mathcal{M}_n(R)$ is an idempotent-simple ring.*

Proof. (1) If $e = 0$, there is nothing to prove. So, we assume $e \neq 0$. Let $f^2 = f \in S := eRe$. Then $RfR = R$, so $SfS = (eRe)f(eRe) = eR(efe)Re = eRfRe = eRe = S$. Hence S is idempotent-simple.

(2) Let E be a nonzero idempotent of $S := \mathcal{M}_n(R)$. We need to show that $S = SES$. By hypothesis, E is similar to a diagonal matrix. Thus, we can assume that $E = \text{diag}(e_1, e_2, \ldots, e_n)$ with each e_i an idempotent and with $e_1 \neq 0$. Then $Re_1R = R$, and this clearly implies that $(RE_{i1})E(RE_{j1}) = (Re_1R)E_{ij} = RE_{ij} \in SES$ for all i, j, and so $SES = S$. □

The following corollary follows since IF rings are idempotent-simple.

Corollary 10. *Let R be an IF ring and $n \geq 1$. Then eRe is idempotent-simple for any $e^2 = e \in R$, and $\mathbb{M}_n(R)$ is an idempotent-simple ring if in addition R is an ID ring.*

Remarks. Is every idempotent-simple ring an IF ring? Note that if there is an IF ring R such that eRe is not IF for some $e^2 = e \in R$, then eRe is an idempotent-simple ring that is not IF. But we do not know whether every corner ring of an IF ring is again IF.

For a square matrix A over a commutative ring, the determinant and the trace of A are denoted by $\det(A)$ and $\text{tr}(A)$, respectively.

Proposition 11. *Let R be a nonzero commutative ring and $n \geq 1$. Then $\mathcal{M}_n(R)$ is idempotent-simple iff R is indecomposable.*

Proof. The necessity is clear. For the sufficiency, let $A = (a_{ij})$ be a nontrivial idempotent in $S := \mathcal{M}_n(R)$. Then $\det(A)^2 = \det(A)$. But $\det(A) \neq 1$, so $\det(A) = 0$. Similarly, $\det(I_n - A) = 0$. Expanding the latter determinant we see that $1 \in \sum_{i \cdot j} Ra_{ij}$. By a standard result in ring theory, we can write $SAS = \mathcal{M}_n(I)$ for some ideal $I \subseteq R$, which must be R (since it contains all a_{ij}). This shows that $SAS = S$, so S is idempotent-simple. □

Remarks. Is $\mathcal{M}_n(R)$ an IF ring for every commutative indecomposable ring R? In view of Proposition 3(2), one can show that, for a commutative ring R, $\mathcal{M}_n(R)$ is IF iff $\mathcal{M}_n(R/\text{nil}(R))$ is IF. Thus, $\mathcal{M}_n(R)$ is IF for any commutative indecomposable ring R iff $\mathcal{M}_n(R)$ is IF for any commutative indecomposable reduced ring R. But we do not know whether $\mathcal{M}_n(R)$ is an IF ring for every commutative indecomposable reduced ring R. We do not even know whether the matrix ring over an IF ring is again IF.

3. On IF Matrix Rings

This section is devoted to the proof of the following theorem.

Theorem 12. *Let n be a positive integer. Over any ring R, the diagonal idempotents $E_{11} + \cdots + E_{ii}$, $i = 1, \ldots, n$, are fine in the matrix ring $\mathcal{M}_n(R)$.*

Proof. In the proof we use determinants. However, as $E_{11} + \cdots + E_{ii}$ $(i = 1, \ldots, n)$ are in $\mathcal{M}_n(\mathbb{Z} \cdot 1_R)$, a subring of $\mathcal{M}_n(R)$, it suffices to show that $E_{11} + \cdots + E_{ii}$ $(i = 1, \ldots, n)$ are fine elements in $\mathcal{M}_n(\mathbb{Z} \cdot 1_R)$. Thus, without loss of generality, we can assume that R is commutative.

Since I_n is fine in $\mathcal{M}_n(R)$, for the proof in the general $n \times n$ case we show that there is a nilpotent T_n such that all $T_n - (E_{11} + \cdots + E_{ii})$, $i = 1, \ldots, n-1$, have the same determinant, namely $(-1)^n$. Hence all diagonal reductions are fine.

We consider some special upper triangular matrices with ± 1 alternating on the diagonal and also on top of the diagonal, that is

$$D_{2k} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}, \quad D_{2k+1} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

We distinguish two cases.

Case 1: $n = 2k+1$. The nilpotent has the block form $T_{2k+1} = \left[\begin{smallmatrix} D_{2k} & \alpha \\ \mathbf{x} & 0 \end{smallmatrix}\right]$ with

$$\alpha = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} (-1)^{k+1} & x_1 & \cdots & x_{2k-1} \end{bmatrix}.$$

Notice that $\mathrm{tr}(T_{2k+1}) = 0$ and the unknowns x_i will be determined such that $\mathrm{tr}(T_{2k+1}^m) = 0$ for $2 \leq m \leq 2k$.

We first show that all $\det(T_{2k+1} - E_{11} - \cdots - E_{ii}) = -1$, for $i = 1, \ldots, 2k$, (for any choice of the unknowns x_i) so all these matrices are units. Equivalently, $E_{11} + \cdots + E_{ii}$, $i = 1, \ldots, 2k$, are fine if we show that T_{2k+1} is (indeed) nilpotent.

We next compute $\det(T_{2k+1} - E_{11})$ by successively reducing it to smaller and smaller minors. We start with

$$\det \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & -1 \\ (-1)^{k+1} & x_1 & x_2 & \cdots & x_{2k-2} & x_{2k-1} & 0 \end{bmatrix}.$$

Since the first row has only one nonzero entry, the determinant equals minus the minor obtained by deleting the first row and the second column, i.e.

$$-\det \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & -1 \\ -1 & x_3 & x_4 & \cdots & x_{2k-2} & x_{2k-1} & 0 \end{bmatrix}$$

and so on, until we reach $(-1)^{k-1} \det \left[\begin{smallmatrix} 0 & -1 \\ (-1)^{k+1} & 0 \end{smallmatrix}\right] = -1$ (entries on the first initial column and the last two rows). Therefore $T_{2k+1} - E_{11}$ is a unit since it has determinant -1.

Notice that in the process of successively reducing the size of the minors (starting with the initial $\det(T_{2k+1} - E_{11})$), we have successively deleted the second, the third and so on columns, ending with but the last column. This shows right away that in this computation the unknowns x_i do not intervene, and, carefully observing the rows (successively) deleted, that also the entries on the (main) diagonal do not intervene. Hence the value of this determinant does not change if we subtract 1 from any of the entries on the diagonal, excepting the last.

Consequently, all determinants $\det(T_{2k+1} - E_{11}) = \det(T_{2k+1} - E_{11} - E_{22}) = \cdots = \det(T_{2k+1} - E_{11} - E_{22} - \cdots - E_{2k,2k}) = -1$ are equal, that is all the possible diagonal reductions are fine (since below we show that the unknowns x_i may be chosen such T_{2k+1} is indeed nilpotent).

What remains is just to choose the unknowns x_i such that $\mathrm{tr}(T^m_{2k+1}) = 0$ for $2 \leq m \leq 2k$ and $\det(T_{2k+1}) = 0$. A block computation shows that

$$x_{2k-1} = x_{2k-2} = k = C^1_k \quad \text{and} \quad x_{2k-3} = -\frac{(k-1)k}{2} = -C^2_k.$$

Finally, the entries of the last row of T_{2k+1} are precisely

$$(-1)^{k+1}; (-1)^{k+1}; (-1)^k C^{k-1}_k; (-1)^k C^{k-1}_k; (-1)^{k-1} C^{k-2}_k;$$
$$\cdots - C^2_k; -C^2_k; C^1_k; C^1_k; 0.$$

It only remains to prove that T_{2k+1} is nilpotent. A hard computation will indeed show that the characteristic polynomial of T_{2k+1} is $p_{T_{2k+1}}(X) = X^{2k+1}$. For the reader's convenience we first give all details for T_7. The characteristic polynomial is obtained by a successive expansion of minors, each of these having only two entries in the first column.

$$\begin{aligned}
&p_{T_7}(X)\\
&\quad = \det(XI_7 - T_7)\\
&\quad = \det\begin{bmatrix} X-1 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & X+1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & X-1 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & X+1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & X-1 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & X+1 & 1\\ -1 & -1 & 3 & 3 & -3 & -3 & X \end{bmatrix}\\
&\quad = (X-1)\det\begin{bmatrix} X+1 & 1 & 0 & 0 & 0 & 0\\ 0 & X-1 & -1 & 0 & 0 & 0\\ 0 & 0 & X+1 & 1 & 0 & 0\\ 0 & 0 & 0 & X-1 & -1 & 0\\ 0 & 0 & 0 & 0 & X+1 & 1\\ -1 & 3 & 3 & -3 & -3 & X \end{bmatrix} + 1\\
&\quad = (X-1)\left[(X+1)\det\begin{bmatrix} X-1 & -1 & 0 & 0 & 0\\ 0 & X+1 & 1 & 0 & 0\\ 0 & 0 & X-1 & -1 & 0\\ 0 & 0 & 0 & X+1 & 1\\ 3 & 3 & -3 & -3 & X \end{bmatrix} + 1\right] + 1
\end{aligned}$$

$$= (X-1)\left\{(X+1)\left[(X-1)\det\begin{bmatrix} X+1 & 1 & 0 & 0\\ 0 & X-1 & -1 & 0\\ 0 & 0 & X+1 & 1\\ 3 & -3 & -3 & X\end{bmatrix}+3\right]+1\right\}+1$$

$$= (X-1)\left\{(X+1)\left[(X-1)\left((X+1)\det\begin{bmatrix} X-1 & -1 & 0\\ 0 & X+1 & 1\\ -3 & -3 & X\end{bmatrix}+3\right)+3\right]+1\right\}+1$$

$$= (X-1)\left\{(X+1)\left[(X-1)\left((X+1)\left\{(X-1)\det\begin{bmatrix} X+1 & 1\\ -3 & X\end{bmatrix}+3\right\}+3\right)+3\right]+1\right\}+1$$

$$= (X-1)\{(X+1)[(X-1)((X+1)\{(X-1)[(X+1)X+3]+3\}+3)+3]+1\}+1$$

$$= (X^2-1)^3X+3(X^2-1)^2(X-1)+3(X^2-1)^2+3(X^2-1)(X-1)$$

$$+3(X^2-1)+X-1+1 = X^7.$$

Now, in the general odd case the last line of the corresponding computation gives $(X^2-1)^k\mathbf{X} + C_k^{k-1}(X^2-1)^{k-1}(X-1) + C_k^{k-1}(X^2-1)^{k-1} + C_k^{k-2}(X^2-1)^{k-2}(X-1) + \cdots + C_k^1(X^2-1)(X-1) + C_k^1(X^2-1) + (X-1) + 1$.

The last step is now easy. We write the first factor $\mathbf{X}$ (the bold style is for reader's convenience) as $1+(X-1)$ and so

$$p_{T_{2k+1}}(X) = [1+(X-1)][(X^2-1)+1]^k = X^{2k+1}.$$

Case 2: $n=2k$. The nilpotent has the block form $T_{2k} = \left[\begin{smallmatrix} D_{2k-1} & \alpha\\ \mathbf{x} & -1\end{smallmatrix}\right]$ with

$$\alpha = \begin{bmatrix} 0\\ \vdots\\ 0\\ 1\end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} (-1)^k & x_1 & \cdots & x_{2k-2} \end{bmatrix}.$$

Notice that $\operatorname{tr}(T_{2k}) = 0$ and the unknowns x_i will be determined such that $\operatorname{tr}(T_{2k}^m) = 0$ for $2 \le m \le 2k-1$.

As in the odd case, we analogously show that all $\det(T_{2k} - E_{11} - \cdots - E_{ii}) = 1$, for $i = 1, \ldots, 2k-1$, (for any choice of the unknowns x_i) so all these matrices are units. Equivalently, $E_{11} + \cdots + E_{ii}$, for $i = 1, \ldots, 2k-1$, are fine if we show that T_{2k} is (indeed) nilpotent.

As in the odd case, it just remains to show how we choose the unknowns x_i in order to have the nilpotent T_{2k}. A block computation shows that

$$x_{2k-2} = -k = -C_k^1, \quad x_{2k-3} = 0 \quad \text{and} \quad x_{2k-4} = \frac{(k-1)k}{2} = C_k^2.$$

Finally, the entries of the last row of T_{2k} are precisely

$$(-1)^k; 0; (-1)^{k-1}C_k^{k-1}; 0; (-1)^{k-2}C_k^{k-2}; 0; \cdots; 0; C_k^2; 0; -C_k^1; -1.$$

An analogous (easier) computation shows that the characteristic polynomial $p_{T_{2k}}(X) = X^{2k}$ and the proof is complete.

For reader's convenience we give below all the details for T_6.

$$p_{T_6}(X) = \det\begin{bmatrix} X-1 & -1 & 0 & 0 & 0 & 0 \\ 0 & X+1 & 1 & 0 & 0 & 0 \\ 0 & 0 & X-1 & -1 & 0 & 0 \\ 0 & 0 & 0 & X+1 & 1 & 0 \\ 0 & 0 & 0 & 0 & X-1 & -1 \\ 1 & 0 & -3 & 0 & 3 & X+1 \end{bmatrix}$$

$$= (X-1)\det\begin{bmatrix} X+1 & 1 & 0 & 0 & 0 \\ 0 & X-1 & -1 & 0 & 0 \\ 0 & 0 & X+1 & 1 & 0 \\ 0 & 0 & 0 & X-1 & -1 \\ 0 & -3 & 0 & 3 & X+1 \end{bmatrix} + 1$$

$$= (X-1)(X+1)\det\begin{bmatrix} X-1 & -1 & 0 & 0 \\ 0 & X+1 & 1 & 0 \\ 0 & 0 & X-1 & -1 \\ -3 & 0 & 3 & X+1 \end{bmatrix} + 1$$

$$= (X-1)(X+1)\left[(X-1)\det\begin{bmatrix} X+1 & 1 & 0 \\ 0 & X-1 & -1 \\ 0 & 3 & X+1 \end{bmatrix} + 3\right] + 1$$

$$= (X-1)(X+1)\{(X-1)[(X+1)((X-1)(X+1)+3)]+3\}+1$$

$$= (X^2-1)^3 + 3(X^2-1)^2 + 3(X^2-1) + 1 = [(X^2-1)+1]^3 = X^6.$$

For completeness, here is the last line of the corresponding computation.

$$\begin{aligned} p_{T_{2k}}(X) &= (X^2-1)^k + C_k^{k-1}(X^2-1)^{k-1} + C_k^{k-2}(X^2-1)^{k-2} \\ &\quad + \cdots + C_k^1(X^2-1) + 1 \\ &= [(X^2-1)+1]^k = X^{2k}. \end{aligned}$$ □

Remarks. (1) The statement that E_{11} is fine in $\mathcal{M}_n(R)$ for any $n \geq 2$ implies Theorem 12. In fact, $E_{11} + \cdots + E_{k+1,k+1} = \begin{bmatrix} I_k & 0 \\ 0 & E_{11} \end{bmatrix}$, where $E_{11} \in \mathcal{M}_{n-k}(R)$ is fine and $I_k \in M_k(R)$ is fine. So $\begin{bmatrix} I_k & 0 \\ 0 & E_{11} \end{bmatrix}$ is fine in $\mathcal{M}_n(R)$. However, perhaps surprising is the fact that all the diagonal matrices $E_{11}, E_{11}+E_{22}, \ldots, E_{11}+E_{22}+\cdots+E_{n-1,n-1}$

can be added to the single nilpotent matrix T_n to become invertible and the matrix T_n can be explicitly constructed.

(2) A legitimate question would be, why we start with the units D_n and not with some simpler units like the identity matrix or with units which have 1's on the secondary diagonal and all the other entries zero.

However, it is easy to check (alternatively use [3]), that I_2 cannot be completed to a 3×3 nilpotent.

Another easy choice would be the $n \times n$ matrices

$$U_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Though U_2 can be nilpotent completed (see [3]), U_3 and U_4 cannot (similar calculation).

Corollary 13. *If every idempotent matrix over R is conjugate to $\left[\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for some k, then every matrix ring over R is IF.*

Following Cohn [5, p. 17], we call a ring R a *projective-free* ring if every finitely generated projective R-module is free of unique rank. By [5, Proposition 4.5, p. 18], a ring R is projective-free precisely when R has invariant basis number and every idempotent matrix over R is conjugate to a matrix of the form $\left[\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Projective-free rings include local rings (see [5, Corollary 5.5, p. 22]). Corollary 13 has a quick consequence.

Corollary 14. *Matrix rings over a projective-free ring (particularly, a local ring) are IF.*

Corollary 15. *Let R be an Abelian ID ring. Then $\mathcal{M}_n(R)$ is an IF ring iff R contains only the trivial idempotents. In particular, matrix rings over an elementary divisor domain are IF.*

Proof. The necessity is clear because $\mathcal{M}_n(R)$ is indecomposable. For the sufficiency, every idempotent matrix over an ID ring is conjugate to a diagonal matrix, and hence is conjugate to $\left[\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for some k (because R contains only the trivial idempotents). □

Corollary 16. *A semiperfect ring R is IF iff R is isomorphic to a matrix ring over a local ring.*

Proof. The sufficiency follows from Corollary 14. For the necessity, since R is semiperfect, idempotents lift modulo $J(R)$ in R. So, by Proposition 3, $R/J(R)$ is

IF, and hence is indecomposable. But, as $R/J(R)$ is semisimple, we deduce that $R/J(R) \cong \mathcal{M}_n(D)$ where $n \geq 1$ and D is a division ring. Because idempotents lift modulo $J(R)$, any set of matrix units of $R/J(R)$ can be lifted to a set of matrix units of R (for details see [9, Theorem 23.10]), and so $R \cong \mathcal{M}_n(S)$ with $S/J(S) \cong D$. □

Corollary 17. *A right self-injective, strongly π-regular ring R is IF iff R is isomorphic to a matrix ring over a local ring.*

Proof. The sufficiency is by Corollary 14. For the necessity, as R is right self-injective and strongly π-regular, $J(R)$ is nil and $R/J(R)$ is a finite direct product of matrix rings over strongly regular rings by [7]. But, as R is IF, $R/J(R)$ is IF by Proposition 3, and hence is indecomposable by Proposition 2. So $R/J(R) \cong \mathcal{M}_n(T)$ where $n \geq 1$ and T is a strongly regular ring. For $\mathcal{M}_n(T)$ to be indecomposable, T cannot have any nontrivial central idempotent. So it follows that T is a division ring. As argued in proving Corollary 16, $R \cong \mathcal{M}_n(S)$ with $S/J(S) \cong T$. □

Proposition 18. *Let R be an integral domain. Consider an idempotent $E = \left[\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right]$, where $bc = a(1-a)$. Then E is conjugate to a diagonal matrix iff there exist $x, y, x_0, y_0 \in R$ such that $a = xy$, $c = xx_0$ and $b = yy_0$.*

Proof. We have replaced our proof, by the one suggested by the referee, which is shorter and more conceptual.

For the sufficiency part, assume that x, y, x_0, y_0 exist. To prove that E is conjugate to E_{11}, it suffices to show that $\mathrm{im}(E)$ or $\ker(E)$ contains a unimodular vector. (Here, of course, we are thinking of $\mathcal{M}_2(R)$ as the endomorphism ring of R^2.) If $a = 0$, $\mathrm{im}(E)$ contains the unimodular vector $(b, 1)^T$ (here T stands for the transpose.) If $a \neq 0$, then $1 - xy = x_0y_0$, so $\ker(E)$ contains the unimodular vector $(y_0, -x)^T$. Conversely, assume E is conjugate to E_{11}. Then R^2 has a basis $\{(y, x_0)^T, (-y_0, x)^T\}$ such that $E\left[\begin{smallmatrix} y & -y_0 \\ x_0 & x \end{smallmatrix}\right] = \left[\begin{smallmatrix} y & 0 \\ x_0 & 0 \end{smallmatrix}\right]$, where we may assume that $xy + x_0y_0 = 1$. Therefore $E = \left[\begin{smallmatrix} y & 0 \\ x_0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} x & y_0 \\ -x_0 & y \end{smallmatrix}\right]$. This gives right away $a = xy$, $b = yy_0$, $c = xx_0$, as desired. □

An integral domain is a GCD domain if every pair a, b of nonzero elements has a greatest common divisor, denoted by $\gcd(a, b)$, i.e. there is a unique minimal principal ideal containing the ideal generated by two given nonzero elements. GCD domains include unique factorization domains, Bézout domains, principal ideal domains and valuation domains. A basic property of a GCD domain is needed for the next corollary: *If a divides bc and* $\gcd(a, b) = 1$ *in a GCD domain, then a divides c.* In fact, $\gcd(a, b) = 1$ implies $\gcd(ac, bc) = c$. As a is a common divisor of ac and bc, a divides $\gcd(ac, bc)$. That is, a divides c.

Corollary 19. *Let R be a GCD domain. Then $\mathcal{M}_2(R)$ is an IF ring.*

Proof. Let $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ be a nontrivial idempotent (the case $a = 0$ is dealt as in the previous proof), i.e. $bc = a(1-a)$ and let $x = \gcd(a, c)$. If $a = xy$ and $c = xx'$ it follows that $\gcd(y, x') = 1$. Since $bx' = y(1-a)$, by the GCD domain hypothesis, y divides b, say $b = yy'$ and the conditions in the previous proposition are fulfilled. □

The next result actually gives an explicit fine decomposition of a nontrivial idempotent 2×2 matrix over a GCD domain (e.g. $\mathcal{M}_2(\mathbb{Z}[t_1, \ldots, t_n])$).

Proposition 20. *Let $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ be a nontrivial idempotent over a GCD domain, $a = xy$, $c = xx'$, $b = yy'$ and $1 - a = x'y'$. A fine decomposition of E is*

$$\begin{bmatrix} (x-x')(y+y') & (y+y')^2 \\ -(x-x')^2 & -(x-x')(y+y') \end{bmatrix} + \begin{bmatrix} -xy' + x'(y+y') & -y^2 - yy' - y'^2 \\ x^2 - xx' + x'^2 & xy' + (x-x')y \end{bmatrix}.$$

Proof. The first term has zero trace and zero determinant and, after computation, the determinant of the second term is $2xx'yy' + x^2y^2 + x'^2y'^2 = 2a(1-a) + a^2 + (1-a)^2 = 1$. Hence, it is indeed a sum of a nilpotent and a unit. □

Remark (suggested by referee). Actually, the right emphasis should not be on the fact that R being a GCD domain implies that $S := M_2(R)$ is IF. Rather, it should be stressed the fact that R being a GCD domain implies that every nontrivial idempotent in S is conjugate to E_{11}. In this case, S is trivially IF, since E_{11} is (obviously) a fine idempotent in S.

Proposition 21. *Let R be an integral domain. Consider a nontrivial idempotent $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ in $\mathcal{M}_2(R)$, where $bc = a(1-a)$. Then E is fine in $\mathcal{M}_2(R)$ iff there exist $p, q, r \in R$ such that $p^2 + qr = 0$ and $(2a-1)p + br + cq \in U(R)$. In particular, if $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is fine, then $(2a-1)R + bR + cR = R$.*

Proof. Suppose E is fine. Then $E = B + U$ where $B \in \mathcal{M}_2(R)$ is nilpotent and $U \in \mathcal{M}_2(R)$ is a unit. Let Q be the field of fractions of R. Then in $\mathcal{M}_2(Q)$, B is similar to sE_{12} for some $s \in Q$. So, $\mathrm{tr}(B) = 0$ and $\det(B) = 0$. So $B = \begin{bmatrix} p & q \\ r & -p \end{bmatrix}$ where $p, q, r \in R$ and $p^2 + qr = 0$. Now, $U = \begin{bmatrix} a-p & b-q \\ c-r & 1-a+p \end{bmatrix}$, so $\det(U) = (2a-1)p + br + cq \in U(R)$. The sufficiency is clear. □

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