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Subgroups generated by images of endomorphisms of Abelian groups and duality

Grigore Călugăreanu, Andrey R. Chekhlov and Piotr A. Krylov

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Abstract. A subgroup H of a group G is called endo-generated if it is generated by endo-images, i.e. images of endomorphisms of G. In this paper we determine the following classes of Abelian groups: (a) the endo-groups, i.e. the groups all of whose subgroups are endo-generated; (b) the endo-image simple groups, i.e. the groups such that no proper subgroup is an endo-image; (c) the pure-image simple, i.e. the groups such that no proper pure subgroup is an endo-image; (d) the groups all of whose endo-images are pure subgroups; (e) the ker-gen groups, i.e. the groups all of whose kernels are endo-generated. Some dual notions are also determined.

1 Introduction

In what follows we use the short term *endo-image* for an endomorphic image of a group.

As early as 1955, Fuchs, Kertesz and Szele determined (see [7]) the Abelian groups *all of whose subgroups are endo-images* denoting by P this property. A torsion group has this property iff all its primary components have this property.

Their results are the following.

Theorem. *A p-group G has property P iff the final rank of G is equal to that of a basic subgroup B of G iff G is a homomorphic image of B.*

Theorem. *A group G with torsion-free rank $r = r_0(G) > 0$ has property P iff*

(i) *in the case $r < \aleph_0$, the group G is of the form $T \oplus \bigoplus_r \mathbb{Z}$, where T is a torsion group with property P (covered by the previous theorem),*

(ii) *in the case $r \geq \aleph_0$, the group contains a free direct summand of rank r and in the torsion subgroup $T(G)$,*

each primary component G_p of final rank $> r$ is a p-group with property P.

Corollary. *A torsion-free group G has property P iff G is either a free group of finite rank or has a free direct summand of rank $|G|$.*

In the same paper the authors notice that for a group G all subgroups are endo-images iff all subgroups that are direct sums of cyclic groups are endo-images.

In particular, *any bounded group has property P* and a *countable p-group fails to have this property iff it has a non-zero divisible part (i.e. direct summands* $\mathbb{Z}(p^{\infty})$*) and its reduced part is bounded.*

In the sequel, the word "group" means always "Abelian group". For a group G, $E =: \mathrm{End}(G)$ denotes the endomorphism ring. For a subset X of a torsion-free group G, $\langle X \rangle_*$ denotes the pure subgroup generated by X (i.e. the intersection of all pure subgroups which include X). For other unexplained notions and notation we refer to [5] and [6].

It is natural to consider the following generalization:

Definition. Let H be a subgroup of a group G and let $E =: \mathrm{End}(G)$. We say that H is *endo-generated* if $H = \sum\{f(G) : f \in E, f(G) \leq H\}$. Obviously, every endo-image of G is endo-generated. In particular, as images of idempotent endomorphisms, direct summands are endo-images and so endo-generated. Further, a group all of whose subgroups are endo-generated will be called an *endo-group*.

However, since every endomorphism $f \in E$ such that $f(G) \leq H$ determines (and is determined by) a homomorphism $f: G \to H$, notice that the above notions may be *equivalently defined* using a more general well-known construction: for groups A and B set $S_A(B) := \sum_{\alpha \in \mathrm{Hom}(A,B)} \alpha(A)$, a fully invariant subgroup of B called the *A-socle of B* (or the *trace* of A in B). If $S_A(B) = B$, we say that B is *A-generated*. Moreover, the group B is called *finitely A-generated* if there exist finitely many homomorphisms $\varphi_i: A \to B$, $i = 1, \dots, n$, such that $B = \sum_{i=1}^{n} \varphi_i(A)$.

Therefore, *a subgroup H of a group G is (finitely) endo-generated iff H is (finitely) G-generated* and a group G is a (finite) endo-group iff all its subgroups are (finitely) G-generated.

The plan of the paper is the following: in Section 2, the endo-groups are completely determined and in Section 3 we determine the endo-image simple groups (i.e. the groups G in which the only endo-images are 0 and G) and then we determine two other classes of groups introducing pure subgroups into our discussion: no pure subgroup is an endo-image, or, all endo-images are pure subgroups.

In Section 4 we determine for groups a notion introduced in [12]: ker-gen groups, that is, groups that generate their kernels (i.e. for every $f \in E$, $\ker(f)$ is endo-generated).

In Section 5, we study some dual notions to those mentioned above. The paper ends with a list of open problems.

2 Endo-groups

Examples. All groups with property P are endo-groups (i.e. all endo-images are endo-generated).

Since all subgroups of $\mathbb{Z}$ are endo-images (of the multiplications), $\mathbb{Z}$ is *more* than an endo-group. So are all simple groups, i.e. $\mathbb{Z}(p)$ for any prime p. Moreover, all cyclic groups are endo-groups and elementary groups are endo-groups.

Note that $\mathbb{Q}$ *is not an endo-group*. Indeed, all proper (which properly include $\mathbb{Z}$) subgroups (i.e. the rational groups) are not endo-generated. Actually *more* holds: $\mathbb{Q}$ has no proper endo-images and the same is true for $\mathbb{Z}(p^\infty)$.

Alternatively (Szele), End(G) is a field iff G is isomorphic to $\mathbb{Q}$ or $\mathbb{Z}(p)$ for some prime p. These have no proper endo-images.

It is easy to check that *if H is a fully invariant subgroup of an endo-group G, then G/H is also an endo-group.*

First notice that G-generated subgroups abound. Indeed, we can prove the following:

Proposition 1. (1) *In any separable group G all pure fully invariant subgroups are G-generated.*

(2) *In any group G all torsion pure subgroups are G-generated.*

Proof. (1) Let $0 \neq x \in F$, where F is a pure fully invariant subgroup of G. Since G is separable, we have $x \in G_1 \oplus \cdots \oplus G_n$, where $r(G_i) = 1$ and $i = 1, \ldots, n$. Then $F' = F \cap (G_1 \oplus \cdots \oplus G_n) = (F \cap G_1) \oplus \cdots \oplus (F \cap G_n)$ and $F \cap G_i = G_i$ if $F \cap G_i \neq 0$. Consequently, F' is a direct summand of G and since $x \in F'$, F is G-generated.

(2) Let H be a torsion pure subgroup of G. Then $H = D \oplus K$, where D is the divisible part of H and K is reduced. Since H is pure in G, so is K and so K is included in some reduced subgroup of G. If $D \neq 0$, then D is a direct summand of G so D is G-generated. The subgroup K is also G-generated since K is generated by cyclic direct summands of G. □

Some more examples are given in the following

Proposition 2. (1) *In any homogeneous torsion-free separable group G, any pure subgroup is G-generated.*

(2) *Let $G = \prod_{i \in I} G_i$, where G_i are torsion-free groups of rank 1 and same type and I is infinite. Then every pure subgroup of G is G-generated iff the type of each G_i is idempotent.*

(3) *Let* $G = \prod_{i \in I} G_i$ *be a vector group, where* G_i *are torsion-free groups of rank* 1 *and type* $\mathbf{t}_i = \mathbf{t}(G_i)$. *Then every pure subgroup of* G *is* G*-generated iff for any* $0 \neq x \in G$ *there exists a type* $\mathbf{t}_i$ *such that* $\mathbf{t}_i \leq \mathbf{t}(x)$.

Proof. (1) According to [6, Proposition 87.2], any pure subgroup of finite rank is a direct summand of G. Whence the conclusion.

(2) If the set I is infinite and the type $\mathbf{t} = \mathbf{t}(G_i)$ is not idempotent, then G has a pure subgroup H of rank 1 and $\mathbf{t}(H) < \mathbf{t}$. Then from [6, Lemma 96.1] it follows that $\mathrm{Hom}\,(G, H) = 0$, i.e. H is not G-generated. Conversely, according to [6, Lemma 96.4] the group G is separable and homogeneous, and we use (1).

(3) Let $H = \langle x \rangle_*$. Since $\mathrm{Hom}\,(G, H) \neq 0$, by [6, Lemma 96.1], $\mathbf{t}_i \leq \mathbf{t}(x)$ for some $\mathbf{t}_i$. Conversely, let F be a pure subgroup of G and $0 \neq x \in F$. Since $\mathbf{t}_i \leq \mathbf{t}(x)$, the corresponding group G_i homomorphic generates $\langle x \rangle_*$. Since every homomorphism $G_i \to \langle x \rangle_*$ lifts to some endomorphism of group G, it follows that F is G-generated. □

In what follows, we completely determine the endo-groups (i.e. the groups all of whose subgroups are endo-generated), a class of groups, larger than the class defined by property P (the groups all of whose subgroups are endo-images), in the Introduction.

We mention that, under the name of *self-generators*, endo-groups were characterized in [9, Theorem 2.1], using equivalences of module categories. For the sake of completeness, we have provided a direct (specific) proof below.

We already noticed that the definition of an endo-generated subgroup of a group G and the definition of a G-generated subgroup are equivalent.

Let A_i, B $(i \in I)$ be groups and let $f_i \colon A_i \to B$ $(i \in I)$ be homomorphisms. If $f \colon \bigoplus_{i \in I} A_i \to B$ is the factorization homomorphism, it is well known that $f(\bigoplus_{i \in I} A_i) = \sum_{i \in I} f_i(A_i)$. Hence $S_A(B) = B$ (i.e. B is A-generated) iff B is an epimorphic image of some A-free group (i.e. an arbitrary direct sum of copies of A), and a group is *finitely* A*-generated* iff an epimorphism $\bigoplus_n A \to B$ exists for some positive integer n.

Since every cyclic group is a homomorphic image of the group $\mathbb{Z}$, we immediately obtain the following:

Lemma 3. *If the group* G *has a direct summand isomorphic to* $\mathbb{Z}$*, then* G *is an endo-group.*

For torsion-free or mixed groups the converse also holds.

Proposition 4. *Let* G *be a torsion-free group or a (genuine) mixed group. Then* G *is an endo-group iff* G *has a direct summand isomorphic to* $\mathbb{Z}$.

Proof. Since one way is already covered by the previous lemma, suppose G is an endo-group and let H be a subgroup of G with $H \cong \mathbb{Z}$. Since H is G-generated, H is an epimorphic image of some G-free group and so a non-zero homomorphism $f: G \to H$ exists with $\operatorname{Im} f \cong \mathbb{Z}$. Hence $G = A \oplus \ker f$, where $A \cong \operatorname{Im} f$, as desired. □

As for torsion groups, it suffices to consider p-groups. Note that if $C \cong \mathbb{Z}(n)$, then $S_C(G) = G[n]$ for any group G.

Proposition 5. *A p-group G is an endo-group iff its reduced part is unbounded or G is a bounded group.*

Proof. To show that the conditions are necessary, assume the opposite, that is $G = B \oplus D$, where $0 \neq D$ is a divisible group and $p^k B = 0$. The group D is isomorphic to a direct sum of groups $\mathbb{Z}(p^\infty)$. Since all non-zero endomorphisms of $\mathbb{Z}(p^\infty)$ are epimorphisms, it follows that the subgroups $G[p^m]$ for any $m > k$ are not G-generated.

Conversely, let $G = A \oplus D$, where A is an unbounded group, D is a divisible group or $D = 0$ and let H be a subgroup of G. For any $k \geq 1$, a decomposition $A = C \oplus B$ exists, where $C \cong \mathbb{Z}(p^n)$ and $n \geq k$. Then clearly $S_G(H) \supseteq H[p^k]$ and so $S_G(H) = H$.

In the remaining case, let G be a bounded group and $0 = p^{k+1}G \neq p^k G$. Then $S_{\mathbb{Z}(p^k)}(H) = H$, whence $S_G(H) = H$, so H is G-generated. □

Since the structure of endo-groups is determined, we can easily deduce the following consequences.

(a) *The endo-group property may not pass to fully invariant subgroups.*

(b) *Arbitrary direct sums of torsion endo-groups are endo-groups.*

(c) *Any group, containing a mixed or torsion-free endo-group as the direct summand is an endo-group.*

3 Three extreme classes

Related to the class of groups considered in [7], one may consider the class of *groups G in which the only endo-images are* 0 *and* G (*endo-image simple*, for short). Obviously, the simple groups, i.e. $\mathbb{Z}(p)$ for any prime p, belong to both classes.

Proposition 6. *A group G is endo-image simple iff $G \cong \mathbb{Z}(p^\infty)$ or $G \cong \mathbb{Z}(p)$ for some prime p or $G \cong \mathbb{Q}$.*

Proof. A group G is endo-image simple iff all non-zero endomorphisms are surjective. From this the claim follows. □

It is also natural to consider *pure subgroups in a similar study* and to determine the groups such that no pure subgroup is an endo-image (*pure-image simple*, for short), or, the groups all of whose pure subgroups are endo-images. This is done in the remainder of this section.

As for *pure-image simple*, we start by observing that since direct summands are pure, such a group must be indecomposable, so cocyclic or torsion-free.

Since a group $G \neq 0$ is pure-simple iff $r(G) = 1$, all cocyclic groups are endo-pure simple and these are the only torsion groups with this property. The same argument works for all rational groups (not only $\mathbb{Q}$) since these are pure-simple.

Since mixed groups are not indecomposable, it remains to determine the pure-image simple torsion-free groups which are indecomposable and of rank at least 2.

First observe that since pure-image simple groups are indecomposable, these cannot contain non-trivial isomorphic pure subgroups. Since in [2] the groups with isomorphic non-trivial pure subgroups were studied, we infer that a pure-image simple group is divisible or reduced. As is well known, any divisible pure-image simple group is isomorphic to $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$ for some prime p. Any reduced pure-image simple group is either torsion, i.e. cyclic p-group for some prime p, or else an indecomposable torsion-free group.

This enables us to describe pure-image simple groups for a large class of groups: the *cotorsion* groups (i.e. groups whose extensions by torsion-free groups split). Recall from [5, (54.4)] that *a torsion group is cotorsion iff it is a direct sum of a divisible group and a bounded group*, and from [5, (54.5)] that *a torsion-free group is cotorsion iff it is algebraically compact* (i.e. G is *algebraically compact* if G is a direct summand in every group H that contains G as pure subgroup). By a theorem of Kaplansky [5, (40.4)], *every non-zero reduced algebraically compact group contains a direct summand isomorphic to J_p (the group of p-adic integers) or $\mathbb{Z}(p^k)$ for some positive integer k and some prime p*. It follows that the directly indecomposable algebraically compact groups are J_p, $\mathbb{Q}$, and the subgroups of $\mathbb{Z}(p^\infty)$. Therefore:

Proposition 7. *A non-zero cotorsion group G is pure-image simple iff G is a divisible pure-image simple group or a cyclic p-group or the group of p-adic integer numbers for some prime p.*

Before proceeding, we recall the following definitions (see [6, Section 92]). Let G and H be torsion-free groups of finite rank such that G is contained in the divisible hull of H. Then G is *quasi-contained* in H if $nG \leq H$ for some positive integer n, and *quasi-equal* to H if also H is quasi-contained in G. A group G

is a *quasi-direct sum* of subgroups $K_1, \dots, K_m$ of its divisible hull if G is quasi-equal to $K_1 \oplus \dots \oplus K_m$.

A group having only trivial quasi-direct decompositions is called *strongly indecomposable*.

In order to determine the remaining pure-image simple torsion-free groups, we first need the following:

Lemma 8. (1) *If G is a pure-image simple torsion-free group, then the type of any of its torsion-free factor-groups of rank* 1 *cannot be the type of any non-zero element of group G.*

(2) *Any pure-image simple torsion-free group G of rank ≤ 3 is strongly indecomposable.*

(3) *Any homogeneous torsion-free group G of rank ≤ 3 is pure-image simple iff it is strongly indecomposable.*

Proof. (1) Indeed, if a factor-group G/A is torsion-free of rank 1 and $\mathbf{t}(G/A) = \mathbf{t}(a)$ for some $0 \neq a \in G$, then the pure subgroup $\langle a \rangle_*$ is an endo-image.

(2) By way of contradiction, assume the group G is quasi-decomposable. Then $nG \leq A \oplus B \leq G$ for some positive integer n, where A is a pure subgroup of rank 1 and B is a pure subgroup of rank 1 or 2 and so $G/B \cong A$ (actually G/B is quasi-isomorphic to A, but since the rank of A is 1, quasi-isomorphisms are isomorphisms). Hence A is an endo-image, a contradiction.

(3) If the factor-group $G/(A \oplus B)$ is not bounded for all pure subgroups A and B of rank 1 and ≤ 2, respectively, then by the homogeneous hypothesis, both types $\mathbf{t}(G/B), \mathbf{t}(G/A) > \mathbf{t}(G)$, i.e. any non-zero pure subgroup rank 1 or rank 2 of G is not an endo-image. That the condition is necessary follows from (2). □

Notice that the converses of (2) and (3) both fail.

Example 9. (1) Strongly indecomposable torsion-free groups G of rank 2 which are not pure-image simple do exist.

(2) Strongly indecomposable not homogeneous pure-image simple torsion-free groups G of rank 2 do exist.

Proof. (1) Let A and B be torsion-free groups of rank 1 such that $\mathbf{t}(A) = \mathbf{t}(B)$ and $pA \neq A$, $pB \neq B$ for some prime number p. Consider a subgroup of the divisible hull of the group $A \oplus B$, namely $G = \langle A \oplus B, p^{-\infty}(a + b) \rangle$, where $0 \neq a \in A$, $0 \neq b \in B$ are fixed elements and $p^{-\infty}(a+b) = \{p^{-1}(a+b), p^{-2}(a+b), \dots\}$. Note that A and B are pure subgroups of G. Indeed, (say) for A, suppose that

$nx = a' \in A$ for some positive integer n and $x \in G \setminus (A \oplus B)$. Then

$$x = s_1x_1 + \cdots + s_mx_m,$$

where $x_1 = p^{-k_1}(a+b), \ldots, x_m = p^{-k_m}(a+b)$. If $k = \max\{k_1, \ldots, k_m\}$, then

$$\begin{aligned} np^k x &= ns_1 p^k x_1 + \cdots + ns_n p^k x_m \\ &= n(s_1 p^{k-k_1} + \cdots + s_m p^{k-k_m})a + n(s_1 p^{k-k_1} + \cdots + s_m p^{k-k_m})b \\ &= p^k a'. \end{aligned}$$

Since all elements in this equality belong to $A \oplus B$,

$$n(s_1 p^{k-k_1} + \cdots + s_m p^{k-k_m})b = 0$$

and so $s_1 p^{k-k_1} + \cdots + s_m p^{k-k_m} = 0$. Hence $x = 0$, since all considered groups are torsion-free. Finally, observe that $\mathbf{t}(G/B) = \mathbf{t}(a + b)$ and so the pure subgroup $\langle a + b\rangle_*$ is an endo-image.

(2) Let A, B be torsion-free groups of rank 1 such that $p_1A = A$, $p_2B = B$ and $p_2A \neq A$, $p_1B \neq B$ for some prime numbers $p_1 \neq p_2$. Now consider a subgroup of the divisible hull of the group $A \oplus B$, namely $G = \langle A \oplus B, p^{-\infty}(a+b)\rangle$, where $0 \neq a \in A$, $0 \neq b \in B$ and p is a some prime number with $pA \neq A$, $pB \neq B$.

Let X be a pure subgroup of rank 1 of the group G. If $A \subseteq X$, then G/X is p-divisible and p_2-divisible. But the group G has no such non-zero elements. Similarly if $B \subseteq X$. If $X \neq A, B$ and $0 \neq x \in X$, then $nx = sa + tb$ for some $n, s, t \in \mathbb{Z}$. Then $sa + X = -tb + X$ so G/X is p_1-divisible and p_2-divisible. But the group G has no such non-zero elements. Consequently, the group G is pure-image simple. □

Moreover, we can provide the following:

Example 10. Pure-image simple torsion-free groups of rank n exist for any positive integer n. For any infinite cardinal number $\mathfrak{m}$, less than the first strongly inaccessible cardinal number, pure-image simple torsion-free groups of power $\mathfrak{m}$ do exist.

Proof. For the first example, take the reduced *cohesive* group (a torsion-free group with only divisible factor groups modulo non-zero pure subgroups; see [6] or [10]) G of rank n. Every $f \in E$ is a monomorphism, so G has no isomorphic non-trivial pure subgroups.

For the second example, take any group A of cardinality $\mathfrak{m}$ from a rigid system (such groups exist according to [6, Theorem 89.2]). Since any endomorphism of A is a multiplication with a rational number, A has no non-trivial pure endo-images. □

Alternatively, for every n there is a rank n group with endomorphism ring isomorphic to $\mathbb{Z}$. For infinite rank there is a whole class of torsion-free groups with endomorphism ring isomorphic to $\mathbb{Z}$. Any of these groups are also examples of pure-image simple groups.

Next recall that a group A is called *sp-group*, if it is a reduced mixed group with infinitely many non-zero p-components A_p, such that the natural embedding $\bigoplus_p A_p \to A$ can be extended to a pure embedding $A \to \prod_p A_p$.

In [1], the following criterion for a group to be a sp-group was established.

Theorem 11. *The following conditions are equivalent for a reduced mixed group A with infinitely many non-zero p-components A_p:*

(1) *A is a sp-group, i.e., the pure embeddings $\bigoplus_p A_p \subset A \subseteq \prod_p A_p$ hold.*

(2) *Both embeddings $\bigoplus_p A_p \subset A \subseteq \prod_p A_p$ hold and $A/(\bigoplus_p A_p)$ is a divisible torsion-free group.*

(3) *For each prime p, there is a group B_p such that $A = A_p \oplus B_p$ with $pB_p = B_p$.*

We can now characterize the third class of groups, *the groups all of whose endo-images are pure subgroups*.

Proposition 12. *In a group G all endo-images are pure subgroups iff G is one of the following groups:*

(1) *G is a divisible group,*

(2) *G is a torsion group and every p-component is elementary or divisible,*

(3) *any p-component of G is elementary or divisible and the reduced part of G is a sp-group.*

Proof. In order to show that the conditions are necessary, we use multiplication by a prime number p as an endomorphism of G. Since pG is (by hypothesis) pure in G, we get $pG = G \cap pG = p(pG)$, i.e. pG is p-divisible. So if $pG_p \neq 0$, then the p-component G_p is divisible. If $pG_p = 0$, then $G = G_p \oplus B_p$ with $pB_p = B_p$ and this completes the proof by Theorem 11.

As for sufficiency, let $H = f(G)$, $f \in E$, be an endo-image of G. Since any p-component G_p is divisible or elementary and $G = G_p \oplus B_p$, where $pB_p = B_p$, any p-component H_p is divisible or elementary, and $H = H_p \oplus C_p$ for some subgroup $C_p \leq H$. If H_p is divisible, then G_p is divisible and $pG = G$ implies $pH = H$. But if H_p is elementary, then so is G_p and $pB_p = B_p$ implies $f(G_p) \cap f(B_p) = 0$. Hence $H = f(G) = f(G_p) \oplus f(B_p)$ and so $C_p \cong f(B_p)$. In particular, C_p is p-divisible. Therefore H_p is p-pure in G and so is C_p since it is p-divisible. Finally, H is pure in G. □

4 Ker-gen groups

The notion studied in this section was suggested by a similar one given in [12] for R-modules.

We say that an R-module M *generates its kernels* (*ker-gen* for short) if for every $f \in \mathrm{End}_R(M)$, $\ker(f)$ is M-generated. This notion was introduced in [12] in order to compare *morphic* modules with modules whose endomorphism ring is *left morphic*. An R-module M is called *morphic* if $M/f(M) \cong \ker(f)$ for every endomorphism $f \in \mathrm{End}_R(M)$, and a ring R is called *left morphic* if ${}_RR$ is morphic.

In the sequel we discuss these notions for $\mathbb{Z}$-modules, that is, for Abelian groups.

Both morphic groups and endo-groups are included in the class of ker-gen groups. However, these two subclasses are not related: $\mathbb{Q}$ is morphic but no endo-group, $\mathbb{Z}$ is endo-group but not morphic.

From [3] we already have a list of ker-gen groups:

(i) A torsion-free group is morphic only if it is divisible.

(ii) A divisible group is morphic only if it is torsion-free. This occurs iff

$$G \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \cdots \oplus \mathbb{Q},$$

i.e. a finite direct sum of $\mathbb{Q}$.

Therefore, the *morphic torsion groups are reduced*.

(iii) A torsion group is morphic iff all its primary components are morphic.

(iv) A (reduced) p-group G is morphic iff it is finite and homogeneous.

(v) The splitting morphic mixed groups are exactly the groups

$$G = T(G) \oplus D(G) = \bigoplus_p (\mathbb{Z}(p^{k_p})^{n_p} \oplus \mathbb{Q}^k$$

with non-negative integers k_p, n_p, and k.

Another known situation when a module M generates its kernels is when $\ker f$ is a direct summand of M for every $f \in \mathrm{End}_R(M)$. We say that M is *kernel-direct* in this case. As is well known, this happens if $\mathrm{End}(M)$ is regular. Then we have another list of ker-gen groups (see [6, Section 112.7]):

(a) If G is a direct sum of a torsion-free divisible group and an elementary group, then it is kernel-direct and so ker-gen.

(b) Elementary groups are kernel-direct and so ker-gen.

For the main result of this section recall that for a set Π of prime numbers, $\mathbb{Q}_\Pi$ denotes the group (ring) of all rational numbers with denominator coprime to all $p \in \Pi$.

Theorem 13. (1) *A torsion group G is ker-gen iff every unreduced p-component G_p has an unbounded reduced part.*

(2) *A divisible group is ker-gen iff it is torsion-free.*

(3) *Let $G = R \oplus D$, where $D = \bigoplus_{p \in \Pi} D_p \oplus D_0$ is the divisible part of G, R is reduced, D_0 is a torsion-free part of D and Π is the set of primes with $D_p \neq 0$. Then G is ker-gen iff the following conditions hold:*

  (i) *R is ker-gen with $p^n R \neq p^{n-1} R$ for all $p \in \Pi$ and every positive integer n,*

  (ii) *if $D_0 \neq 0$ there is a subgroup H of R such that the factor-group R/H is torsion-free of rank 1 and $\mathbf{t}(R/H) \leq \mathbf{t}(\mathbb{Q}_\Pi)$,*

  (iii) *the kernel of any homomorphism $R \to D$ is R-generated.*

Proof. (1) Assume that G_p has a direct summand isomorphic to $\mathbb{Z}(p^\infty)$. Since there are epimorphisms of $\mathbb{Z}(p^\infty)$ with kernels of arbitrary large order, it follows that G_p has cyclic direct summands of arbitrary large order, i.e. the reduced part of G_p is unbounded. Conversely, the statement follows since all subgroups of G are cyclic direct summands of G.

(2) Obvious, since homomorphic images of divisible groups are divisible.

(3) If G is ker-gen, clearly R is ker-gen and the kernel of any homomorphism $R \to D$ is R-generated. Since $\mathbb{Z}(p^\infty)$ has endomorphisms with kernels of arbitrary large order, it follows that $p^n R \neq p^{n-1} R$ for all $p \in \Pi$ and positive integers n. Finally, since $\mathbb{Q}/\mathbb{Q}_\Pi \cong \bigoplus_\Pi \mathbb{Z}(p^\infty)$, we have $\mathrm{Hom}\,(R, \mathbb{Q}_\Pi) \neq 0$ and so such a subgroup H exists.

Conversely, first notice that every endomorphism of the group G is determined by three homomorphisms (in an upper triangular matrix form): an endomorphism of R, a homomorphism $R \to D$ and an endomorphism of D. In view of conditions on the group R, it only remains to show that the kernel of every endomorphism of D is G-generated. Every endomorphism of the group D is also determined by three homomorphisms: an endomorphism of the group D_0, a homomorphism $D_0 \to T(D)$ and an endomorphism of the group $T(D)$. The kernel of the endomorphism of D_0 is a direct summand of D_0. Let D_0/F be isomorphic to some subgroup of $T(D)$. Then $pF = F$ for every prime p not in Π and so F is a $\mathbb{Q}_\Pi$-module. Hence F is generated by homomorphic images of the group $\mathbb{Q}_\Pi$. Finally, since $p^n R \neq p^{n-1} R$ for all $p \in \Pi$ and positive integers n, $R/p^n R$ contains the cyclic groups of order p^n, the whole group $T(D)$ is R-generated. □

Kernels of endomorphisms of any reduced torsion-free group are closed pure subgroups. We mention that in [4] groups in which every closed pure subgroup is a direct summand were studied. Such groups are clearly ker-gen.

Lemma 14. *Let G be a separable torsion-free group and let Ω be the set of types of rank 1 direct summands of G.*

(i) *Every pure subgroup of G is G-generated iff for any $\mathbf{t}_1, \mathbf{t}_2 \in \Omega$ there is $\boldsymbol{\tau} \in \Omega$ with $\boldsymbol{\tau} \leq \mathbf{t}_1, \mathbf{t}_2$.*

(ii) *Assume that for any $\mathbf{t}_1, \mathbf{t}_2 \in \Omega$ there is $\boldsymbol{\tau}' \in \Omega$ with $\boldsymbol{\tau}' \geq \mathbf{t}_1, \mathbf{t}_2$. Then G is ker-gen iff or any $\mathbf{t}_1, \mathbf{t}_2 \in \Omega$ there is $\boldsymbol{\tau} \in \Omega$ with $\boldsymbol{\tau} \leq \mathbf{t}_1, \mathbf{t}_2$.*

Proof. (i) Let $G = G_1 \oplus A = G_2 \oplus B$, where $r(G_1) = r(G_2) = 1$, and $\mathbf{t}_1 = \mathbf{t}(G_1)$ and $\mathbf{t}_2 = \mathbf{t}(G_2)$ are incomparable. Then $G_2 \subseteq G(\mathbf{t}_2) = \{g \in G : \mathbf{t}(g) \geq \mathbf{t}_2\}$ and $G(\mathbf{t}_2) = (G(\mathbf{t}_2) \cap G_1) \oplus (A \cap G(\mathbf{t}_2))$. Since $G(\mathbf{t}_2) \cap G_1 = 0$, $G(\mathbf{t}_2) \subseteq A$ follows and so $G_1 \oplus G_2$ is a direct summand of G. If $0 \neq x \in G_1, 0 \neq y \in G_2$, then since $\langle x + y\rangle_*$ is G-generated, $\boldsymbol{\tau} \leq \mathbf{t}(x + y) < \mathbf{t}_1, \mathbf{t}_2$ for some $\tau \in \Omega$.

Conversely, let H be a pure subgroup of G and $x \in H$. Since G is separable, $\mathbf{t}(x) = \mathbf{t}_1 \cap \cdots \cap \mathbf{t}_n$ for some $\mathbf{t}_1, \ldots, \mathbf{t}_n \in \Omega$. From the hypothesis it follows that there exists $\boldsymbol{\tau} \in \Omega$ with $\boldsymbol{\tau} \leq \mathbf{t}_1 \cap \cdots \cap \mathbf{t}_n$. If A is a direct summand of rank 1 and $\mathbf{t}(A) = \boldsymbol{\tau}$, then $S_A(\langle x\rangle_*) = \langle x\rangle_*$. Each homomorphism $A \to \langle x\rangle_*$ lifts to an endomorphism of the group G, mapping any complement of A into 0. Consequently, H is G-generated.

(ii) Let $0 \neq x \in G_1$, $0 \neq y \in G_2$, where $G_1 \oplus G_2$ is a direct summand of G and $\mathbf{t}(G_1)$, $\mathbf{t}(G_2)$ are incomparable (see (i)). Take $F = (G_1 \oplus G_2)/\langle x + y\rangle_*$, so that $r(F) = 1$. Since $\mathbf{t}(G_3) > \mathbf{t}_1, \mathbf{t}_2$ for some direct summand G_3 of rank 1, there is a homomorphism $g \colon F \to G_3$ and we have $G = G_1 \oplus G_2 \oplus C \oplus G_3$. Define the endomorphism f as follows: $f \mid (G_1 \oplus G_2) = g$, $f \mid C = 1_C$ and $f(G_3) = 0$. Then $\ker(f) = \langle x + y\rangle_* \oplus G_3$. Since $\ker(f)$ is G-generated, $\tau \leq \mathbf{t}(x + y) < \mathbf{t}_1, \mathbf{t}_2$ for some $\boldsymbol{\tau} \in \Omega$. We are done now using (i). □

Proposition 15. (1) *If $G = G_1 \oplus G_2$, where G_1 and G_2 are torsion-free groups of rank 1, then G is a ker-gen group. If the types $\mathbf{t}(G_1)$ and $\mathbf{t}(G_2)$ are incomparable, then G has pure subgroups which are not G-generated.*

(2) *If $G = G_1 \oplus G_2 \oplus \mathbb{Q}$, where G_1 and G_2 are torsion-free groups of rank 1 of incomparable types, then G is not ker-gen.*

Proof. (1) If $0 \neq H \neq G$ is a kernel and $H \neq G_1, G_2$, then $H = \langle x + y\rangle_*$ for some $x \in G_1$ and $y \in G_2$. Since G/H is isomorphic to some subgroup of G, $\mathbf{t}(G/H) \leq \mathbf{t}(G_1)$ or $\mathbf{t}(G/H) \leq \mathbf{t}(G_2)$. On the other hand $\mathbf{t}(G/H) \geq \mathbf{t}(G_1), \mathbf{t}(G_2)$, whence $\mathbf{t}(G_1) \leq \mathbf{t}(G_2)$ or $\mathbf{t}(G_2) \leq \mathbf{t}(G_1)$. But then we have $\mathbf{t}(G_1) = \mathbf{t}(x + y)$ or $\mathbf{t}(G_2) = \mathbf{t}(x + y)$ so G_1 or G_2 generates H. Now if the types $\mathbf{t}(G_1)$ and $\mathbf{t}(G_2)$ are incomparable then $\mathbf{t}(x + y) < \mathbf{t}(G_1), \mathbf{t}(G_2)$, so H is not G-generated.

(2) Follows from the previous lemma. □

Since the structure of ker-gen groups was clarified above, it follows that the *ker-gen property does not pass to (fully invariant) summands*, as the property of being morphic does. As an example, let T be an unbounded p-group. Then by Proposition 5, $T \oplus \mathbb{Z}(p^\infty)$ is an endo-group. However if T_1 is a bounded direct summand of T then the group $T_1 \oplus \mathbb{Z}(p^\infty)$ is not ker-gen.

5 Duality

In [8] a property (denoted Q) which is dual to the property P, recalled in the Introduction, was studied.

Notice that P (every subgroup is an endo-image) is equivalent to the property that every subgroup is isomorphic to some factor group.

A group G has property Q if every homomorphic image of G can be (isomorphically) embedded in G. In other words, every factor group of G is isomorphic to some subgroup of G.

The following results were proved.

Theorem. *An abelian p-group G has property Q iff it contains a direct summand of the form $\bigoplus_{\mathfrak{m}} \mathbb{Z}(p^\infty)$, where $\mathfrak{m} = \min_n \operatorname{rank}(p^n G)$ is the final rank of G.*

Corollary. *A reduced abelian p-group has property Q iff it is bounded.*

Theorem. *An abelian group G of infinite torsion free rank r has property Q iff*

(i) *$r \leq \mathfrak{p}_i$ holds for the final rank $\mathfrak{p}_i$ of the p_i-component G_{p_i} of the torsion subgroup $T(G)$, for each prime p_i,*

(ii) *G contains a direct summand of the form $\bigoplus_r \mathbb{Q} \oplus \bigoplus_i \bigoplus_{\mathfrak{p}_i} \mathbb{Z}(p_i^\infty)$.*

Theorem. *A group G of finite torsion-free rank r has property Q iff*

$$G = F \oplus T = F \oplus \bigoplus_{i=1}^{\infty} T_i,$$

where

(i) *every T_i is a p_i-group of infinite final rank $\mathfrak{p}_i$,*

(ii) *every T_i has property Q,*

(iii) *F is a torsion free group of rank r,*

(iv) *$F = R(\sigma_1) \oplus \cdots \oplus R(\sigma_r)$, where $R(\sigma_i)$ are rational groups of type σ_i satisfying $\sigma_1 \geq \cdots \geq \sigma_r$.*

We first recall a dual construction to the A-socle, namely, the *A-radical* (or *co-trace*) of B: for two groups A and B,

$$K_B(A) := \bigcap_{\alpha \in \operatorname{Hom}(A,B)} \ker \alpha,$$

which is also a fully invariant subgroup of A. If $K_B(A) = 0$, we say that A is *B-cogenerated.*

Let A, B_i $(i \in I)$ be groups and let $f_i \colon A \to B_i$ $(i \in I)$ be homomorphisms. If $f \colon A \to \prod_{i \in I} B_i$ is the factorization homomorphism, it is well known that $\ker(f) = \bigcap_{i \in I} \ker(f_i)$. Hence $K_B(A) = 0$ (i.e. A is B-cogenerated) iff A embeds in a power of B (i.e. a direct product of copies of B).

A group G has the property Q if for every subgroup H, G/H embeds in G, that is, iff there is a homomorphism $f \colon G/H \to G$ with $\ker(f) = 0$ (i.e. monic).

The generalization we deal with in this section is the following: a factor group G/H is *G-cogenerated* if

$$K_G(G/H) := \bigcap_{f \in \operatorname{Hom}(G/H,G)} \ker f = 0.$$

As seen above, G/H is G-cogenerated iff G/H embeds in a power of G iff there is a monomorphism from G/H to a direct product of copies of G.

Dual to endo-groups (determined in Section 2), a group G will be called *co-endo-group* if all its factor groups are G-cogenerated.

In the sequel we determine the co-endo-groups.

Since any torsion-free group has torsion homomorphic images, it follows that *any co-endo-group is a torsion or a mixed group*.

Proposition 16. *A torsion group G is a co-endo-group iff every p-component G_p is bounded or not reduced.*

Proof. If G_p is unbounded, then $G_p/H \cong \mathbb{Z}(p^\infty)$ for some subgroup H. Since a non-zero homomorphism $\mathbb{Z}(p^\infty) \to G_p$ does exist, it follows that G_p is not reduced. The converse is obvious. □

Theorem 17. *A mixed group G is a co-endo-group iff the group G has subgroups isomorphic to $\mathbb{Z}(p^\infty)$, for every prime number p.*

Proof. If the factor group $G/T(G)$ is not p-divisible, then any group of order p^n is a homomorphic image and so every p-component G_p is unbounded. Therefore

$G_p/H \cong \mathbb{Z}(p^\infty)$ for some subgroup H, whence G has a direct summand isomorphic to $\mathbb{Z}(p^\infty)$. In the remaining case, if $G/T(G)$ is p-divisible, again the group G has a direct summand isomorphic to $\mathbb{Z}(p^\infty)$.

The converse follows from the fact that G contains a subgroup isomorphic to $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}(p^\infty)$, which is a coimage group. □

6 Open problems

As we did in the Section 3 for pure subgroups, we may consider fully invariant subgroups instead. Since sums of fully invariant subgroups are fully invariant, *the groups G in which the G-generated subgroups are fully invariant* are precisely the groups in which endomorphic images are fully invariant. These groups were studied in [11].

The converse problem could be of interest:

Problem 1. Describe the groups G in which the fully invariant subgroups are G-generated.

A continuation for Section 4 could be:

Problem 2. Describe the (torsion-free) vector ker-gen groups.

Homogeneous torsion-free completely decomposable groups of finite rank and reduced algebraically compact groups G have the following property: the pure subgroups which are endomorphic images are the direct summands of G. Therefore we may add:

Problem 3. Describe the groups G in which all pure subgroups which are endomorphic images are direct summands of G.

Related to the classes described in Section 3 we also state:

Problem 4. Describe the groups in which every pure subgroup is an endomorphic image (or else, is a kernel of endomorphism).

Recall that a pure subgroup of a torsion group G, which is a direct sum of cyclic groups, is an endomorphic image of G ([5, Section 36.2]).

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Author information

Grigore Călugăreanu, Department of Mathematics,
Babeş-Bolyai University, Cluj-Napoca, Romania.
E-mail: calu@math.ubbcluj.ro

Andrey R. Chekhlov, Faculty of Mechanics and Mathematics,
Tomsk State University, Tomsk, Russia.
E-mail: cheklov@math.tsu.ru

Piotr A. Krylov, Faculty of Mechanics and Mathematics,
Tomsk State University, Tomsk, Russia.
E-mail: krylov@math.tsu.ru