Abelian groups have/are near Frattini subgroups

Simion Breaz, Grigore Călugăreanu

Abstract. The notions of nearly-maximal and near Frattini subgroups considered by J.B. Riles in [20] and the natural related notions are characterized for abelian groups.

Keywords: nearly maximal subgroup, near Frattini subgroup, non-near generator, DA-groups, NDA-groups, Richman type, functorial subgroup, Frattini subgroup

Classification: 20K27

1. Introduction

A subgroup M of an infinite group G is called *nearly maximal* if it is a maximal element of the set of all subgroups of G having infinite index, i.e., if the index $|G : M|$ is infinite but every subgroup of G properly containing M has finite index in G. This concept was introduced by Riles [20] in 1969. Moreover, in the same paper the following notions are defined: $g \in G$ is called *non-near generator* if for every subgroup H of G, $G/(H + \langle g \rangle)$ finite implies G/H finite. If we denote by $\lambda(G) \overset{\text{def}}{=} \{g \in G | g \text{ non-near generator}\}$, and $\mu(G) \overset{\text{def}}{=} \bigcap\{H \leq G | H \text{ nearly maximal in } G\}$, these are two characteristic subgroups of G called the *lower* respectively *upper near Frattini*, such that $\lambda(G) \subseteq \mu(G)$. A group G has a *near Frattini subgroup* (denoted $\Psi(G)$) if $\lambda(G) = \mu(G)$.

After some 30 years of relative silence, it seems that lately these notions have received the attention they probably would have deserved (see Allenby [1], [2], Azarian [3], [4], [5], [6], Franciosi, de Giovani [16]; see also Lennox, Robinson [19]).

In all the previous papers written on this topic, with few exceptions, no reference on the abelian group case, except the finitely generated groups, is made.

In the sequel we shall call *DA-groups* the abelian groups which have a **d**ually **a**tomic lattice of subgroups (i.e. each proper subgroup is contained in a maximal subgroup) and *NDA-group* the abelian groups such that every infinite index subgroup is contained in a nearly-maximal subgroup.

The second author was supported by Kuwait University, Research Project No. SM09/01.

Notice that in [20], the DA-groups are called $\mathfrak{E}$-groups, and the NDA-groups are called $\mathfrak{E}'$-groups and it is proved that each NDA-group is a DA-group.

In the first section we prove that each abelian group has a near Frattini subgroup and we characterize the nearly-maximal subgroups. Actually, it easily turns out that the near Frattini subgroup defines a radical on abelian groups (mainly denoted by $R_{\mathbb{Z}}$), so that results obtained by several authors can be used (and there is a vast bibliography on this topic). We characterize some of the situations that may occur: $\Psi(G) = 0$, $\Psi(G) = T(G)$ the torsion part, respectively $\Psi(G) = G$ and we show that the Frattini subgroup $\Phi(G) \subseteq \Psi(G)$.

In the attempt of characterizing the NDA-groups it occurred that a characterization of the DA-groups would be useful. These are obtained in Sections 2 and 3.

Finally, resuming a result of Dlab ([9]) who proved that every abelian group is the Frattini subgroup of a suitable chosen abelian group, we prove the corresponding 'near-version'.

Our main results can be summarized as follows.

1. Abelian groups have a near Frattini subgroup.
2. Let G be an infinite abelian group and H a subgroup of G. The subgroup H is nearly maximal in G if and only if there is an infinite order element $a \in G - H$ such that $G = H \oplus \langle a \rangle$.
3. A group G is a DA-group if and only if all its p-components are bounded and $G/T(G)$ is of finite torsion-free rank and of reduced Richman type.
4. The near Frattini subgroup always includes the Frattini subgroup.
5. A group is an NDA-group if and only if it is finitely generated.
6. Every abelian group is the near Frattini subgroup of a suitably chosen abelian group.
7. Except for trivial cases, there are no 'minimal' chosen groups in 6.

For finite groups, $\lambda(G) = \mu(G) = \Psi(G) = G$, so this case may be discarded in the sequel. In this note, 'group' will mean 'infinite abelian group'. For unexplained terminology and facts, we refer to [17].

2. Abelian groups have near Frattini subgroups

In the proof of the main result of this section we use the following elementary results.

If $A \leq B$ and $H \cap B = 0$ hold for three subgroups of a group G, then $(H \oplus B)/(H \oplus A) \simeq B/A$.

If $\mathrm{ord}(g) = \infty$ and $K \cap \langle g \rangle = 0$ then G/K has infinite order elements.

Theorem 2.1. *Abelian groups have a near Frattini subgroup.*

Proof: Assume $g \in \mu(G) - \lambda(G)$ is not a non-near generator, i.e., there exists a subgroup $H \leq G$ such that G/H is infinite and $G/(H + \langle g \rangle)$ is finite. Therefore owing to $G/(H+\langle g\rangle) \simeq (G/H)/((H+\langle g\rangle)/H)$, $(H+\langle g\rangle)/H \simeq \langle g\rangle/(H\cap\langle g\rangle)$ must be also infinite. Hence $\mathrm{ord}(g)=\infty$ and $H\cap\langle g\rangle = 0$.

Let K be $\langle g\rangle$-high in G such that $H \leq K$. We prove that K must be nearly maximal in G.

Take $K < N \leq G$. Then G/N is finite (and hence nothing has to be proved), or $N \cap \langle g\rangle \neq 0$ and there is a $k \in \mathbb{N}^*$ such that $kg \in N$. From $H < N$ and $kg \in N$ we infer $H + \langle kg\rangle \leq N$ and $|G/N| \leq |G/(H+\langle kg\rangle)|$. Moreover, from $G/(H+\langle g\rangle) \simeq (G/(H+\langle kg\rangle))/((H+\langle g\rangle)/(H+\langle kg\rangle))$, with finite $G/(H+\langle g\rangle)$ and $(H+\langle g\rangle)/(H+\langle kg\rangle)$, we derive that $G/(H+\langle kg\rangle)$ is finite and hence again G/N is finite. Thus K has to be nearly-maximal.

But then $g \in K$, a contradiction which shows that g is a non-near generator, as desired (i.e., $\mu(G) \subseteq \lambda(G)$). □

Next we characterize the nearly maximal subgroups.

Theorem 2.2. *Let H be a subgroup of a group G. The subgroup H is nearly maximal in G if and only if there is an infinite order element $a \in G - H$ such that $G = H \oplus \langle a\rangle$.*

Proof: First observe that H is nearly maximal in G if and only if G/H has only finite proper homomorphic images. An easy exercise (see e.g. [15]) shows that this is equivalent to $G/H \simeq \mathbb{Z}$ and hence, equivalent to H being a direct summand of G with infinite cyclic complement. □

Thus, *H is nearly maximal in G if and only if $G/H \simeq \mathbb{Z}$.*

Corollary 2.1. *For every group G, $T(G) \leq \Psi(G)$ holds.*

Proof: If G has no nearly-maximal subgroups, $\Psi(G) = G$. If H is nearly-maximal in G then $G = H \oplus \langle g\rangle$ with $\mathrm{ord}(g) = \infty$. Hence $T(G) = T(H \oplus \langle g\rangle) = T(H) \oplus T(\langle g\rangle) = T(H)$. Thus $T(G) \subseteq H$. □

Corollary 2.2. *$\Psi(G)$ is pure in G.*

Proof: The torsion part $T(G)$ being pure in G it suffices to show (we use the previous corollary) that $G/\Psi(G)$ is torsion-free. Let $g \in G - \Psi(G)$ (thus $g \neq 0$ and $\mathrm{ord}(g) = \infty$) and suppose $\mathrm{ord}_{G/\Psi(G)}(g + \Psi(G)) = n \in \mathbb{N}^*$. Then $0 \neq ng \in \Psi(G)$. Using again the proof of Theorem 2.1, $0 \neq ng \in K$ and so $0 \neq ng \in K \cap \langle g\rangle$, a contradiction. □

Corollary 2.3 (of proof). *$G/\Psi(G)$ is torsion-free.* □

In [20], it was mentioned that $\lambda(G)$ and $\mu(G)$ (and hence $\Psi(G)$) are characteristic subgroups of G. In our case more can be proved:

Corollary 2.4. *$\Psi(G)$ is a fully invariant subgroup of G.*

PROOF: Set $S = \{g \in G | \langle g \rangle$ is a direct summand in $G\}$. For an arbitrary $g \in S$ if $N_g = \bigcap\{N | N \oplus \langle g \rangle = G\}$ then according to [17] N_g is a fully invariant subgroup of G. Hence $\Psi(G) = \bigcap_{g \in S} N_g$ is fully invariant too. □

Consequences:

Nearly maximal subgroups are proper direct summands.

$\Psi(G) = G$ holds for non-cyclic indecomposable groups.

$\Psi(G) = G$ if and only if G has no infinite cyclic direct summands.

Torsion groups have no nearly-maximal subgroups.

$\Psi(G) = G$ holds for torsion and for divisible groups.

Let H, N be subgroups of G and $H \nsubseteq N$. If N is nearly-maximal in G then $H \cap N$ is nearly-maximal in H.

Examples. $\mathbb{Z}$ has an unique nearly-maximal subgroup: 0. Hence $\Psi(\mathbb{Z}) = 0$.

All the non-cyclic subgroups H of $\mathbb{Q}$ have $\Psi(H) = H$.

Recall (e.g., from [17]): let $\mathcal{X}$ be a class of groups. If with every $G \in \mathbf{Ab}$ we associate the subgroup $R_{\mathcal{X}}(G) = \bigcap_{\varphi \in \mathrm{Hom}(G,X), X \in \mathcal{X}} \ker \varphi$, $R_{\mathcal{X}} : \mathbf{Ab} \to \mathbf{Ab}$ is a functor, $R_{\mathcal{X}}(G)$ is a functorial subgroup and $R_{\mathcal{X}}$ is a radical (the largest radical with $R_{\mathcal{X}}(X) = 0$ for every $X \in \mathcal{X}$). $R_{\mathcal{X}}$ is uniquely determined by its annihilator class $\mathcal{A}_{R_{\mathcal{X}}} = \{G | R_{\mathcal{X}}(G) = 0\}$, the smallest class containing $\mathcal{X}$ which is closed under formation of products and subgroups. As a special case, if $\mathcal{X} = \{X\}$ is a singleton class, the radical is also called *singly generated* and its annihilator class $\mathcal{A}_{R_X} = \{G | R_X(G) = 0\}$ is easily seen to be the class of all subgroups of products of copies of X (in the general case, for a class $\mathcal{X}$, a group X is called *residually-$\mathcal{X}$* if $\bigcap\{U \leq X | X/U \in \mathcal{X}\} = 0$ and this is equivalent to X being a subgroup of a product of $\mathcal{X}$-groups).

Taking $\mathcal{X} = \{\mathbb{Z}\}$, we obtain at once $R_{\mathbb{Z}}(G) = \Psi(G)$, a functorial subgroup and the corresponding radical Ψ (i.e., $\Psi(G/\Psi(G)) = 0$). Hence, *if H is a subgroup of G then* $\Psi(H) \leq \Psi(G)$ and Ψ *commutes with direct sums.*

Consequences:

$\Psi(G) = 0$ if and only if G embeds in a direct product of infinite cyclic groups.

For every free group F, $\Psi(F) = 0$.

$G/\Psi(G)$ is isomorphic with a subgroup of a direct product of infinite cyclic groups.

(Stein [23]) If G is countable then $G/\Psi(G)$ is free (and hence $\Psi(G)$ is a direct summand with a free complement).

If G is a direct sum of indecomposable torsion-free groups such that $\Psi(G) = 0$ then G is free.

If G is a finite rank torsion-free group such that $\Psi(G) = 0$ then G is free.

([20]): **(a)** for any group homomorphism $f : G \to G'$, $f(\Psi(G)) \leq \Psi(f(G))$; **(b)** $N \leq \Psi(G)$ implies $\Psi(G/N) = \Psi(G)/N$; **(c)** $\Psi(G/N) = 0$ implies $\Psi(G) \leq N$.

$\Psi(G) = T(G)$ if and only if $G/T(G)$ embeds in a direct product of infinite cyclic groups.

If $G/T(G)$ is free then $\Psi(G) = T(G)$ (e.g., finitely generated groups).

Finally let us point out that the problem of the commutation of the near Frattini radical with *direct products*, even if apparently solved in the affirmative by Riles in [21], depends heavily upon the set-theoretic environment (see [10], for a comprehensive exposition), and actually remains true only in $V = L$ — the axiom of constructibility (and many other models without $\aleph_m$ — first measurable cardinal). This result can be related with a question raised by Charles [7] and repeated in [17, vol. 1, Problem 7]: determine if a (pre)radical commutes with (direct) products. Notice that in [14], it is shown that, for a torsion-free group X, R_X commutes with *countable* products if and only if X is a stout group (see Göbel [18]).

See also [11] and [8].

Comparison with the Frattini subgroup.

Obviously, if $G = H \oplus N$ and $\{N_i\}_{i\in I}$ are subgroups of N then $\bigcap\limits_{i\in I}(H \oplus N_i) = H \oplus (\bigcap\limits_{i\in I} N_i)$. Therefore

Proposition 2.1. *$\Phi(G) \subseteq \Psi(G)$ holds.*

Proof: Let H be an arbitrary nearly maximal subgroup in G. Then $G = H \oplus \langle g\rangle$, for a suitable infinite order element $g \in G$. For every prime p, $H \oplus \langle pg\rangle$ is a maximal subgroup in G and $\bigcap\limits_{p\in\mathbb{P}}(H \oplus \langle pg\rangle) = H$. Hence $\Phi(G) \subseteq H$ and so $\Phi(G) \subseteq \Psi(G)$. □

Consequences:

([20]) Every non-generator actually is a non-near-generator.

$\Psi(G)/\Phi(G)$ is a torsion group.

For every divisible group G, $\Psi(G) = G$.

3. DA-groups

In this section we determine the abelian groups (hereafter called *DA-groups*) which have a dually atomic lattice of subgroups (i.e. each proper subgroup is contained in a maximal subgroup).

Lemma 3.1. *For a group G the following conditions are equivalent:*

(a) *G is a DA-group;*

(b) *for every nonzero subgroup H of G, the quotient group G/H is reduced;*

(c) *for every nonzero subgroup H of G, the quotient group G/H is not divisible.* □

Lemma 3.2. *A DA-p-group is a bounded direct sum of cyclics.*

Proof: If B is a basic subgroup of a p-group G and $B \neq G$ then G/B is divisible and we contradict the previous lemma. Hence $G = B$ is a direct sum of cyclics. Moreover, the basic subgroup being unique, this p-group must be bounded. □

For a finite rank torsion-free group G, if $S = \{x_1, x_2, \dots, x_n\}$ is a maximal independent subset of G, and F the (free) subgroup generated by S then G/F is torsion. Recall that the *Richman type* of G is defined as the equivalence class of G/F under quasi-isomorphism of torsion groups. One can write $G/F = \bigoplus_{p \in \mathbb{P}} T_p$ with $T_p = \mathbb{Z}_{p^{i_{p,1}}} \oplus \mathbb{Z}_{p^{i_{p,2}}} \oplus \cdots \oplus \mathbb{Z}_{p^{i_{p,n}}}$ and $0 \leq i_{p,1} \leq i_{p,2} \leq \cdots \leq i_{p,n} \leq \infty$.

If G/F is reduced, we shall say that G is of *reduced* Richman type.

Proposition 3.1. *The DA torsion-free groups are exactly the finite rank groups whose Richman type is reduced.*

Proof: First of all, G has finite rank. Indeed, if for an arbitrary (possibly mixed) group $G \neq T(G)$ the torsion-free rank $r_0(G) = \infty$ then $G/T(G)$ would be an infinite rank torsion-free quotient, with a (divisible) quotient isomorphic to $\mathbb{Q}$.

Further, if G/F would not be reduced, it should have a divisible direct summand and so G would have a divisible proper quotient, contradicting the DA property.

Conversely, let C be a subgroup of G such that G/C is divisible. For a free subgroup F such that G/F is torsion and reduced, G/F is a finite direct sum of bounded p-groups. Together with G/F, also $G/(F+C)$ has to be reduced. But together with G/C, $G/(F+C)$ has to be divisible so that $G/(F+C) = 0$ or $G = F + C$.

Finally, $G/C = (F+C)/C \simeq F/(F \cap C)$, being divisible and finitely generated, must be zero and hence $G = C$.

Therefore G is DA, having no divisible proper quotients. □

Theorem 3.1. *A group G is a DA-group if and only if all its p-components are bounded and $G/T(G)$ is of finite torsion-free rank whose Richman type is reduced.*

Proof: Using the previous results the conditions are necessary. Indeed, take F as above in the definition of the Richman type. Then G/F is torsion and Lemma 3.2 applies.

Conversely, the groups described above are DA because they have only non-divisible quotients. □

4. NDA-groups

Definition. Call *NDA-group* a group G, if every infinite index subgroup of G is contained in a nearly-maximal subgroup.

Lemma 4.1. *A torsion group is NDA if and only if it is finite.*

Proof: Torsion groups have no nearly-maximal subgroups. Hence the only NDA torsion groups are (trivially) the groups with no infinite index subgroups. Thus these groups have to be finite (again [15]). The converse is trivial. □

Proposition 4.1. *A torsion-free group G is an NDA-group if and only if G is free of finite rank.*

Proof: Using [20, Section 6], G must be a DA-group, and hence of finite rank and reduced Richman type.

First notice that G/F cannot be infinite: indeed, G being NDA, F would be included in a nearly-maximal subgroup N and (by Theorem 2.2) as G/N is torsion-free, we contradict G/F is torsion.

Finally, if F is free and G/F is finite, G is free too (e.g. [17, §18.3]).

Conversely, a free group of finite rank is finitely generated. Hence it is NDA, by [20, Proposition 1]. □

Proposition 4.2. *G is an NDA-group if and only if $G/T(G)$ is free of finite rank and $T(G)$ is finite.*

Proof: Obviously homomorphic images of NDA-groups are also NDA-groups. If G is an NDA-group then $G/T(G)$ is also NDA, and, by the previous proposition, is free of finite rank. Hence G is a splitting mixed group and $G = T(G) \oplus F$, with F free of finite rank.

Finally, $T(G) \simeq G/F$ must be NDA, and hence finite, by a previous lemma.

Conversely, if G/H is infinite we have to find a nearly-maximal subgroup $M \leq G$ which includes H.

First observe that in our hypothesis $G/(T(G)+H)$ is also infinite. Indeed, this follows from $G/(T(G)+H) \simeq (G/H)/((H+T(G))/H)$ and $(H+T(G))/H \simeq T(G)/(H \cap T(G))$, a finite group together with $T(G)$.

Further $|G/(T(G)+H)| = |(G/T(G))/((H+T(G))/T(G))|$ being infinite, by hypothesis (and the previous proposition), there is a nearly-maximal subgroup $M/T(G)$ in $G/T(G)$ such that $(H+T(G))/T(G) \leq M/T(G)$. Consequently $H \subseteq H+T(G) \subseteq M$, $|G/T(G) : M/T(G)| = |G : M|$ is infinite, and M is nearly-maximal in G. □

Hence

Theorem 4.1. *G is an NDA-group if and only if G is finitely generated.* □

Example ([3]). $G = \mathbb{Z}(p_1^n) \times \mathbb{Z}(p_2^n) \times \dots$ with countable many primes $p_1, p_2, \dots$ and n a natural number is a DA group but not a NDA group.

5. Abelian groups as near Frattini subgroups

Let us call *surjective* a radical R defined on abelian groups, such that for every group G there exists a group H such that $R(H) = G$.

Obviously $R_{\mathbb{Q}}(G) = T(G)$, the torsion part, is not a surjective radical.

In [9], V. Dlab proved that the Frattini subgroup (on abelian groups) defines a surjective radical (i.e., every abelian group G is the Frattini subgroup of a suitable chosen abelian group H). In his proof, the formula $\Phi(H) = \bigcap_{p\in\mathbb{P}} pH$ is used together with the "inverse" construction $H = \sum_{p\in\mathbb{P}} (p^{-1}G)$ in a divisible envelope $\overline{G}$ of G (actually, $H/G = S(\overline{G}/G)$).

In what follows we show that an analogous result holds for near Frattini subgroups of abelian groups, i.e., the near Frattini subgroup defines a surjective radical.

It is not simple to find a group H such that $\Psi(H) = \mathbb{Z}$. Such a group was constructed (using an idea of Shelah [22]) by Eklof in [13], and generalized by Eda in [12]: let X be a subset of $2^\omega = \{\text{functions } x : \omega \to 2 = \{0,1\} \subset \mathbb{Z}\}$ of cardinality $\aleph_1$ and let Y be the finite restrictions of X, i.e., $y \in Y$ if there exists $x \in X$ and $n \in \omega$ such that $y = x|_n$ with $n = \{0, 1, \ldots, n-1\}$. Let V be the vector space over $\mathbb{Q}$ with basis $X \cup Y \cup \{e\}$ (e a new symbol), i.e., $V = (\bigoplus_{x\in X} \mathbb{Q}x) \oplus (\bigoplus_{y\in Y} \mathbb{Q}y) \oplus (\mathbb{Q}e)$ and $A \subseteq V$ be generated as $\mathbb{Z}$-module by $X \cup Y \cup \{e\}$ plus all elements of the form $(x - (x|_n) - x(n)e)/p_n$ where $x \in X$, $n \in \omega$ and p_n is the n-th prime.

Then A is $\aleph_1$-free, of cardinality $\aleph_1$, not separable and $\Psi(A) = R_{\mathbb{Z}}(A) = \mathbb{Z}$.

Another construction is given in [10]: if $0 \to \mathbb{Z} \to A \to \mathbb{Z}^\kappa \to 0$, $\kappa < \aleph_m$ is an extension of infinite order then $R_{\mathbb{Z}}(A) = \mathbb{Z}$ (moreover, if $X \leq G$ and $R_X(G/X) = 0$ then $R_X(G) = X$ if and only if $S : 0 \to X \to G \to G/X \to 0$ has infinite order as element of $\operatorname{Ext}(G/X, X)$, $\operatorname{End}(X)$-module, see [14]).

Proposition 5.1. *Let R be a radical on abelian groups such that there is a group A such that $R(A) = \mathbb{Z}$. Then R is surjective.*

Proof: Indeed, if for an arbitrary group, $G = F/N$ is the presentation by generators and relations, then $G = F/N = (\bigoplus \mathbb{Z})/N = (\bigoplus R(A))/N = R(\bigoplus A)/N = R((\bigoplus A)/N)$. □

Corollary 5.1. *Each (abelian) group is the near Frattini subgroup of a suitable (abelian) group.* □

Further on, it would have been natural to try to prove that two 'minimal' previously chosen groups are unique up to isomorphism, that is, if H_1 and H_2 contain G, $\Psi(H_1) = \Psi(H_2) = G$ and for every $K_i < H_i$, $i \in \{1, 2\}$, $\Psi(K_i) \neq G$ then $H_1 \simeq H_2$.

Unfortunately, in this direction, the analogy stops.

First of all, observe that not every divisible envelope $\overline{G}$ of a group G contains a subgroup H such that $\Psi(H) = G$. Indeed, for all the subgroups $R \leq \mathbb{Q}$ such that $\mathbb{Z} \leq R$ we have $\Psi(R) = R$ if $R \not\cong \mathbb{Z}$ and $\Psi(\mathbb{Z}) = 0$ (i.e. $\Psi(R) \not\cong \mathbb{Z}$).

Further, consider $\mathcal{H}_G = \{X \text{ group } | \Psi(X) = G\}$ naturally ordered by inclusion. If for a group G we have $\Psi(G) = G$, obviously G is minimal in $(\mathcal{H}_G, \subseteq)$ (indeed, $K \leq H$ implies $\Psi(K) \leq \Psi(H)$).

In what follows we prove that excepting these groups G, *there are no minimal elements in* $(\mathcal{H}_G, \subseteq)$.

Recall that the finite order elements are obviously non-near generators.

Lemma 5.1. *An infinite order element $g \in G$ is a non-near generator if and only if for every subgroup H of G, $\langle g \rangle \cap H = 0$ implies $G/(\langle g \rangle + H)$ is infinite.*

Proof: If g is an infinite order element and $\langle g \rangle \cap H = 0$ then $\langle g \rangle / (\langle g \rangle \cap H) \simeq (H + \langle g \rangle)/H$ and also G/H are infinite. Thus $G/(\langle g \rangle + H)$ is also infinite (otherwise, g being non-near generator, $G/(\langle g \rangle + H)$ finite, would imply G/H finite).

Conversely, owing to our hypothesis, we only need to prove that $G/(\langle g \rangle + H)$ finite implies G/H finite for an element g such that $\langle g \rangle \cap H \neq 0$ (because if $\langle g \rangle \cap H = 0$, $G/(\langle g \rangle + H)$ is not finite). In this case, $\langle g \rangle / (\langle g \rangle \cap H) \simeq (H + \langle g \rangle)/H$ are finite so that the required implication holds (indeed, $G/(\langle g \rangle + H) \simeq (G/H)/((H + \langle g \rangle)/H))$. □

Lemma 5.2. *If H is minimal in $(\mathcal{H}_G, \subseteq)$, for every subgroup $H' < H$ which is comparable with G, $|H : H'|$ is infinite.*

Proof: If $G \leq H'$ there is an element $g \in \Psi(H) - \Psi(H') = G - \Psi(H')$ (notice that $g \notin \Psi(H')$ implies $\operatorname{ord}(g) = \infty$), i.e., for every subgroup U of H, such that $\langle g \rangle \cap U = 0$, $H/(\langle g \rangle + U)$ is infinite, and, there exists a subgroup V of H', such that $\langle g \rangle \cap V = 0$ and $H'/(\langle g \rangle + V)$ is finite. Hence (because $V \leq H' < H$) also $H/(\langle g \rangle + V)$ is infinite and then $|H : H'|$ is infinite (notice that $|H : (\langle g \rangle + V)| = |H : H'| \cdot |H' : (\langle g \rangle + V)|$).

For a subgroup $H' < G$ we apply the previous case. □

Proposition 5.2. *If H is minimal in $(\mathcal{H}_G, \subseteq)$ then H/G is divisible.*

Proof: Indeed, all the proper subgroups H'/G of H/G have infinite index, because $|(H/G) : (H'/G)| = |H : H'|$.

□

Corollary 5.2. *If $\Psi(G) \neq G$ then $(\mathcal{H}_G, \subseteq)$ has no minimal elements.*

Proof: If $\Psi(H) = G$ then $H/G = \Psi(H/G) = \Psi(H)/G = 0$ and so $H = G$ and $\Psi(G) = G$. □

Hence, there is not a ψ-closure to be defined by the near Frattini subgroup $\Psi(G)$ of a group G.

Finally we list some open problems:

(1) The characterization of the groups G with $G/\Psi(G)$ free.

(2) In which conditions on a torsion-free group G is $\Psi(G)$ balanced ?

(3) The characterization of the groups G with $\Psi(G)/\Phi(G)$ elementary (see [3], finitely generated).

References

[1] Allenby R.B.J.T., *Some remarks on the upper Frattini subgroup of a generalized free product*, Houston J. Math. **25** (1999), no. 3, 399–403.

[2] Allenby R.B.J.T., *The existence and location of the near Frattini subgroup in certain generalized free products*, Houston J. Math. **26** ((2000)), no. 3, 463–468.

[3] Azarian M.K., *On the near Frattini subgroups of certain groups*, Houston J. Math. **20** (1994), no. 3, 555–560.

[4] Azarian M.K., *On the near Frattini subgroup*, Group Theory (Granville, OH, 1992), pp. 21–29, World Sci. Publishing, River Edge, NJ, 1993.

[5] Azarian M.K., *On the near Frattini subgroup of the generalized free product of finitely generated nilpotent groups*, Houston J. Math. **23** (1997), no. 4, 613–615.

[6] Azarian M.K., *On the near Frattini subgroups of amalgamated free products with residual properties*, Houston J. Math. **23** (1997), no. 4, 603–612.

[7] Charles B., *Sous-groupes fonctoriels et topologies*, Studies on Abelian Groups, Paris, 1968, pp. 75–92.

[8] Cutler D.O., Fay T.H., *Radicals of abelian groups*, Comm. Algebra **18** (2) (1990), 403–411.

[9] Dlab V., *The Frattini subgroups of abelian groups*, Czechoslovak Math. J. **10** (1960), 1–16.

[10] Dugas M., Göbel R., *On radicals and products*, Pacific J. Math. **118** (1) (1985), 79–104.

[11] Dugas M., Oxford E.P., *Preradicals induced by torsion-free abelian groups*, Comm. Algebra **17** (4) (1989), 981–1002.

[12] Eda K., *Cardinality restrictions on preradicals*, Abelian Group Theory (Perth, 1987), Contemp. Math., 87, Amer. Math. Soc., Providence, RI, 1989, pp. 277–283.

[13] Eklof P.C., *Set theoretical methods in homological algebra and abelian groups*, Les Presses de l'Université de Montréal, 1980.

[14] Fay T.H., Oxford E.P., Walls G.L., *Singly generated socles and radicals*, Abelian Group Theory, Lecture Notes Math. 1006, Springer, 1983, pp. 671–684.

[15] Fedorov Yu.S., *On infinite groups in which all non-trivial subgroups have finite index*, Uspekhi Mat. Nauk **1/6** (1957), 187–189.

[16] Franciosi S., de Giovanni F., *On groups with many nearly maximal subgroups*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **9** (1998), no. 1, 19–23.

[17] Fuchs L., *Infinite Abelian Groups, vol. 1 and 2*, Academic Press, New York, London, 1970, 1973.

[18] Göbel R., *On stout and slender groups*, J. Algebra **35** (1975), 39–55.

[19] Lennox J.C., Robinson D.J.S., *Nearly maximal subgroups of finitely generated soluble groups*, Arch. Math. Basel **38** (1982), 289–295.

[20] Riles J.B., *The near Frattini subgroups of infinite groups*, J. Algebra **12** (1969), 155–171.

[21] Riles J.B., *The near Frattini subgroups of direct products of groups*, J. London Math. Soc. (2) **8** (1974), 239–244.

[22] Shelah S., *On uncountable abelian groups*, Israel J. Math. **32** (4) (1979), 311–330.
[23] Stein K., *Analytische Functionen mehrer komplexer Veränderlichen zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem*, Math. Ann. **123** (1951), 201–222.

Faculty of Mathematics and Computer Sci., "Babeş-Bolyai" University, Romania
E-mail: bodo@math.ubbcluj.ro

Department of Mathematics and Computer Sci., Faculty of Science, Kuwait University, Kuwait
E-mail: calu@mcs.sci.kuniv.edu.kw

(*Received* November 26, 2001, *revised* May 20, 2002)