RINGS WHOSE CLEAN ELEMENTS ARE UNIQUELY CLEAN

GRIGORE CĂLUGĂREANU AND YIQIANG ZHOU

Abstract. The rings whose every element is uniquely clean are completely characterized in [16]. These rings belong to a larger class of rings for which every clean element is uniquely clean. The latter is the topic of this article.

1. Introduction

Throughout, rings are associative with identity. An element in a ring is *clean* if it is a sum of an idempotent and a unit, *strongly clean* if it is a sum of an idempotent and a unit that commute, and *uniquely clean* if it is uniquely the sum of an idempotent and a unit. A ring is called clean (resp., uniquely clean) if every element is clean (resp., uniquely clean). As a special class of clean rings, uniquely clean rings are completely characterized ([1, 16]). There are several classes of rings, which naturally include uniquely clean rings, whose structures are fully understood (see [10, 11, 12]). This paper is devoted to a new generalization of uniquely clean rings, called CUC rings. Here a ring is called a *CUC* ring if every clean element is uniquely clean. The same line of thought is seen in a work of partially unit-regular rings (rings in which every regular element is unit-regular) in [4] and a work of RS rings (rings in which every regular element is strongly regular) in [20].

The paper displays a new picture in which uniquely clean rings fit. In Section 2, basic properties and various examples of CUC rings are presented. In contrast to uniquely clean rings, CUC rings can occur as polynomial rings. In Section 3,

Key words and phrases. idempotent, unit, clean element, uniquely clean element, abelian ring, semipotent ring, potent ring, Boolean ring.

characterizations of semipotent CUC rings are proved, from which known characterizations of uniquely clean rings can be readily obtained. In particular, it is showed that uniquely clean rings are exactly the potent CUC rings. Section 4 gives a few families of CUC group rings. Section 5 deals with the question of when every factor ring of a ring is CUC, with an answer for a commutative or semipotent ring.

For a ring R, we denote by $J(R)$, $U(R)$, $\mathrm{idem}(R)$ and $\mathrm{nil}(R)$ the Jacobson radical, the unit group, the set of idempotents and the set of nilpotents of R, respectively. We write $\mathrm{cn}(R)$, $\mathrm{scn}(R)$ and $\mathrm{ucn}(R)$ for the set of clean elements, the set of strongly clean elements and the set of uniquely clean elements of R, respectively. The ring of $n \times n$ matrices and the ring of $n \times n$ upper triangular matrices over R are denoted by $\mathbb{M}_n(R)$ and $\mathbb{T}_n(R)$, respectively. For a subring S of a ring R, we do not insist that $1_S = 1_R$. An element in a ring is called 2-*good* if it is the sum of two units.

2. Examples and basic properties

A ring is called a *CUC* ring if every **c**lean element is **u**niquely **c**lean, and a ring is called a *UUC* ring if every **u**nit is **u**niquely **c**lean. Note that the question of when $\mathrm{ucn}(R) \subseteq \mathrm{scn}(R)$ holds true, was addressed in [7]. A ring is *abelian* if each of its idempotents is central.

Proposition 2.1. [8] *The following are equivalent for a ring R:*

(1) *R is CUC.*
(2) $\mathrm{scn}(R) \subseteq \mathrm{ucn}(R)$.
(3) *R is abelian and UUC.*
(4) *R is abelian and $(U(R) + U(R)) \cap \mathrm{idem}(R) = 0$.*

Proof. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). The equivalences are proved in [8, Proposition 27].

(1) $\Rightarrow$ (2). The implication is obvious.

(2) $\Rightarrow$ (3). Every unit of R is strongly clean, so is uniquely clean by (2). Hence R is UUC. For $e^2 = e \in R$, $e = (1-e)+(2e-1)$ is a strongly clean decomposition. So e is uniquely clean by (2), and hence is central by [12, Lemma 2.4]. Hence R is abelian. $\square$

One can easily check that the ring $\mathbb{T}_2(\mathbb{Z}_2)$ is UUC but not abelian. For any non-trivial ring R and any $n \geq 2$, $\mathbb{T}_n(R)$ and $\mathbb{M}_n(R)$ are not CUC.

Examples 2.2. (1) *An indecomposable ring R is CUC if and only if R has the trivial idempotents only and 1 is not 2-good. In particular, $\mathbb{Z}$, $\mathbb{F}_2[t]$ and $\mathbb{F}_2[[t]]$ are CUC.*

(2) *Suppose that R has the trivial idempotents only. Then R is UUC if and only if 1 is not 2-good.*

An indecomposable UUC ring may contain non-trivial idempotents; for example, $\mathbb{T}_2(\mathbb{Z}_2)$. A subring of a uniquely clean ring need not be uniquely clean; for instance, $\mathbb{Z}_2[[t]]$ is uniquely clean but $\mathbb{Z}_2[t]$ is not. This is in contrast to Proposition 2.3(1).

Proposition 2.3. (1) *Let S be a subring of a ring R. If R is CUC (resp., UUC), then so is S.*

(2) *Let $R = \prod R_i$ be a direct product of rings $\{R_i\}$. Then R is CUC (resp., UUC) if and only if so are R_i for all i.*

Proof. In view of Proposition 2.1, it suffices to show the UUC case.

(1) Let $u = v + e$ where $u, v \in U(S)$ and $e \in \mathrm{idem}(S)$. Then $u - (1_R - 1_S) = [v - (1_R - 1_S)] + e$ with $u - (1_R - 1_S), v - (1_R - 1_S) \in U(R)$. Since R is UUC, it follows that $u = v$ and $e = 0$. So S is UUC.

(2) The necessity is by (1). For the sufficiency, let $(u_i) = (v_i) + (e_i)$ where $(e_i) \in \mathrm{idem}(R)$ and $(u_i), (v_i) \in U(R)$. Then, for each i, $u_i = v_i + e_i$ with $u_i, v_i \in U(R_i)$ and with $e_i \in \mathrm{idem}(R_i)$. Since R_i is UUC, it follows that $u_i = v_i$ and $e_i = 0$. So $(u_i) = (v_i)$ and $(e_i) = 0$. Hence R is UUC. $\square$

Corollary 2.4. *Any subdirect product of CUC (resp., UUC) rings is CUC (resp., UUC).*

A ring is called *I-finite* if it contains no infinite orthogonal family of idempotents.

Corollary 2.5. *Let R be an I-finite ring. Then R is CUC if and only if $R = \prod_{i=1}^{n} R_i$ where R_i has the trivial idempotents only and 1 is not 2-good in R_i for all i.*

Proof. The sufficiency follows from Examples 2.2 and Proposition 2.3. For the necessity, being I-finite, R as a right R-module has the ACC on direct summands (see [15, Lemma B.6, page 255]), so $R = \prod_{i=1}^{n} R_i$ where each R_i is indecomposable. By Proposition 2.3, R_i is CUC, so R_i has the trivial idempotents only and 1 is not 2-good in R_i by Examples 2.2. $\square$

A ring R is called *semiperfect* if $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$. It is known that every semiperfect ring is I-finite.

Corollary 2.6. *Let R be a semiperfect ring. Then R is CUC if and only if $R = \prod_{i=1}^{n} R_i$ where $R_i/J(R_i) \cong \mathbb{Z}_2$ for all i.*

For a subring C of a ring D, the set $\mathcal{R}[D, C] := \{(d_1, \ldots, d_n, c, c, \ldots) : d_i \in D, c \in C, n \geq 1\}$, with addition and multiplication defined componentwise, is a ring.

Corollary 2.7. *The ring $\mathcal{R}[D, C]$ is CUC (resp., UUC) if and only if so is D.*

Lemma 2.8. *Let $I \subseteq J(R)$ be an ideal of R such that idempotents lift modulo I. The following hold:*

(1) *R is abelian if and only if R/I is abelian and idempotents lift uniquely modulo I.*
(2) *R is UUC if and only if R/I is UUC.*

Proof. (1)($\Rightarrow$). If $\overline{e} \in \mathrm{idem}(R/I)$, then we can assume that $e \in \mathrm{idem}(R)$. So e is central and so is $\overline{e}$. Assume that $e - f \in I$ where $e, f \in \mathrm{idem}(R)$. Since e, f are central, $(e-f)^3 = e - f$. It follows from $e - f \in I$ that then $e = f$. Hence idempotents lift uniquely modulo I.

(1)($\Leftarrow$). Let $e \in \mathrm{idem}(R)$. Then $e = (1-e)+(2e-1)$, a sum of an idempotent and a unit. Assume that $e = f + u$ where $f^2 = f$ and $u \in U(R)$. Then $\overline{e} = \overline{f} + \overline{u} = \overline{1-e} + \overline{2e-1}$, which are two clean decompositions. Since $\overline{e}$ is central, it is uniquely clean by [12, Lemma 2.4]. It follows that $\overline{f} = \overline{1-e}$. Hence $f = 1 - e$. So e is uniquely clean and hence central by [12, Lemma 2.4].

(2)($\Rightarrow$). Assume that $\overline{u} = \overline{e} + \overline{v}$ where $\overline{u}, \overline{v} \in U(R/I)$ and $\overline{e} \in \mathrm{idem}(R/I)$. We can assume that $e^2 = e$, and so $u = e + (v + j)$ for some $j \in I$. Note that $u, v \in U(R)$ and $j \in J(R)$, so $u = e + (v+j)$ is a clean decomposition. Since R is UUC, $e = 0$, so $\overline{e} = \overline{0}$. Hence R/I is UUC.

(2)($\Leftarrow$). Assume that $u = e + v$ where $u, v \in U(R)$ and $e \in \mathrm{idem}(R)$. Then $\overline{u} = \overline{e} + \overline{v}$ with $\overline{u}, \overline{v} \in U(R/I)$ and $\overline{e} \in \mathrm{idem}(R/I)$. It follows that $\overline{e} = \overline{0}$, so $e \in I$. Thus $e = 0$, and hence R is UUC. $\square$

Corollary 2.9. *Let $I \subseteq J(R)$ be an ideal of R such that idempotents lift modulo I. The following are equivalent:*

(1) *R is CUC.*
(2) *R/I is CUC and idempotents lift uniquely modulo I.*
(3) *R is abelian and R/I is CUC.*

Proof. It is by Proposition 2.1 and Lemma 2.8. $\square$

Remark 2.10. (1) The proof of "(2)($\Leftarrow$)" of Lemma 2.8 shows that if R/I is UUC and $I \subseteq J(R)$, then R is UUC.
(2) It follows from (1) and Proposition 2.1 that if R is abelian and R/I is CUC where $I \subseteq J(R)$, then R is CUC.

Since idempotents lift modulo any nil ideal, Lemma 2.8 and Corollary 2.9 have the following immediate consequences.

Corollary 2.11. *Let I be a nil ideal of R. Then*

(1) *R is UUC if and only if R/I is UUC.*
(2) *R is CUC if and only if R is abelian and R/I is CUC.*

Proposition 2.12. *Let $R = S + I$, where S is a subring of R with $1_R \in S$ and I is an ideal of R, such that $S \cap I = 0$.*

(1) *R is UUC if and only if $(U(R) + U(R)) \cap \mathrm{idem}(I) = 0$ and S is UUC.*
(2) *R is CUC if and only if R is abelian, S is UUC, and eIe is UUC for all $e^2 = e \in I$.*

Proof. (1) For the necessity, R UUC implies that S is UUC by Proposition 2.3, and also that $(U(R) + U(R)) \cap idem(I) = 0$. For the sufficiency, let $u = e + v$ where $u, v \in U(R)$ and $e \in \mathrm{idem}(R)$. Write $u = u_s + u_i, e = e_s + e_i, v = v_s + v_i$ where $u_s, e_s, v_s \in S$ and $u_i, e_i, v_i \in I$. It follows that $u_s = e_s + v_s$ where $u_s, v_s \in U(S)$ and $e_s \in \mathrm{idem}(S)$. Since S is UUC, we have $e_s = 0$, so $e \in I$. From $(U(R) + U(R)) \cap \mathrm{idem}(I) = 0$, it follows that $e = 0$.

(2) For the necessity, R CUC implies that S and eIe are CUC by Proposition 2.3, and that R is abelian by Proposition 2.1. For the sufficiency, it suffices to show that $(U(R)+U(R)) \cap \mathrm{idem}(I) = 0$ because of (1) and Proposition 2.1. Let $u = e+v$ where $u, v \in U(R)$ and $e \in \mathrm{idem}(I)$. Since e is central in R, $eu = e+ev$ with $eu, ev \in U(eIe)$. Since eIe is UUC, it follows that $e = 0$. □

The unique cleanness of a trivial extension and an ideal-extension has been discussed in [9, 16, 17]. Here we consider when the two classical ring constructions are CUC (resp., UUC).

For a bimodule V over a ring A, the *trivial extension* $A \propto V$ of A by V is defined to be the additive abelian group $A \propto V = A \oplus V$ with multiplication $(a, v)(b, w) = (ab, aw + vb)$. For a subset B of A and a subset W of V, we write $B \propto W$ for $\{(b, w) \in A \propto V : b \in B \text{ and } w \in W\}$.

Corollary 2.13. *Let A, B be rings, V a bimodule over A, M an (A, B)-bimodule and $k \geq 1$.*

1. *A is CUC (resp., UUC) if and only if $A[[t]]$ is CUC (resp., UUC).*
2. *A is CUC (resp., UUC) if and only if $A[[t]]/(t^k)$ is CUC (resp., UUC).*
3. *$A \propto V$ is UUC if and only if A is UUC.*
4. *$A \propto V$ is CUC if and only if A is CUC and $ev = ve$ for all $e \in \mathrm{idem}(A)$ and $v \in V$.*
5. *The formal triangular matrix ring $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is UUC if and only if A, B are UUC.*

Proof. (1) Let $R = A[[t]]$, $S = A$ and $I = tR$. Since $\mathrm{idem}(I) = 0$, (1) follows from Proposition 2.12.

(2) Let $R = A[[t]]/(t^k)$, $S = A$ and $I = tR$. Since $\mathrm{idem}(I) = 0$, (2) follows from Proposition 2.12.

(3) and (4) Let $R = A \propto V$, $S = A \propto 0$ and $I = 0 \propto V$. Since $\mathrm{idem}(I) = 0$, (3) and (4) follow from Proposition 2.12.

(5) Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, $S = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $I = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$. Since $\mathrm{idem}(I) = 0$, (5) follows from Proposition 2.12. $\square$

Corollary 2.14. *The following hold:*

1. *Let R be a ring and T be a subring of $R[[t]]$ with $R \subseteq T \subseteq R[[t]]$ (e.g., $T = R[t]$). Then R is CUC (resp., UUC) if and only if so is T.*
2. *The ring $\mathbb{T}_n(R)$ is UUC if and only if R is UUC.*

Proof. (1) It follows from Corollary 2.13(1) and Proposition 2.3(1).

(2) It follows from Corollary 2.13(5). $\square$

One easily sees that a proper matrix ring can not be UUC.

Let A be a ring and let V be an (A, A)-bimodule which is a non-unital ring in which $(vw)a = v(wa)$, $(va)w = v(aw)$ and $(av)w = a(vw)$ hold for all $v, w \in V$ and $a \in A$. Then the *ideal-extension* $\mathbb{I}(A; V)$ of A by V is defined to be the additive abelian group $\mathbb{I}(A;V) = A \oplus V$ with multiplication $(a,v)(b,w) = (ab, aw + vb + vw)$. Note that if S is a ring and $S = A \oplus V$ where A is a subring and $V \lhd S$, then $S \cong \mathbb{I}(A;V)$.

Corollary 2.15. *The ring $S := \mathbb{I}(A,V)$ is CUC if and only if S is abelian, A is UUC, and eVe is UUC for all $e^2 = e \in V$.*

Proof. Let $R = \mathbb{I}(A,V)$, $S = (A,0)$, and $I = (0,V)$. The claim follows from Proposition 2.12(2). □

3. Semipotent CUC rings

A ring R is called *semipotent* if every one-sided ideal not contained in $J(R)$ contains a nonzero idempotent. A semipotent ring R is called *potent* if idempotents lift modulo $J(R)$. Semipotent rings and potent rings were also named as I_0-rings and I-rings, respectively, by Nicholson in [14]. Semipotent CUC rings form a natural class of rings as characterized below, from which new characterizations of uniquely clean rings are obtained. The center of a ring R is denoted by $C(R)$. Following [7, Definition 3.4], we let $\mathrm{ucn}_0(R) = \{e + j : e^2 = e \in C(R), j \in J(R)\}$.

Theorem 3.1. *Let R be a semipotent ring. The following are equivalent:*

(1) *$R/J(R)$ is UUC.*
(2) *$R/J(R)$ is Boolean.*
(3) *$U(R) = 1 + J(R)$.*
(4) *$U(R) \subseteq \mathrm{ucn}_0(R)$.*
(5) *For each $a \in U(R)$, there exists a unique $e^2 = e$ such that $a - e \in J(R)$.*
(6) *For each $a \in U(R)$, there exists $e^2 = e$ such that $a - e \in J(R)$.*

Proof. (1) ⇒ (2). Since R is semipotent, $\overline{R} := R/J(R)$ is semipotent (indeed, potent). We show that $\overline{R}$ is reduced. Assume that $x^2 = 0$ where $0 \neq x \in \overline{R}$. Then, by [13, Theorem 2.1] (or see [19, Remarks 15.5(3), page 134]), there exists $0 \neq y^2 = y \in \overline{R}$ such that $y\overline{R}y \cong \mathbb{M}_2(S)$ for a non-trivial ring S. But one easily sees that $\mathbb{M}_2(S)$ is not UUC: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Thus, $y\overline{R}y$ is not UUC. This is a contradiction to Proposition 2.3(1). Hence $\overline{R}$ is reduced and hence abelian $\big($since for any $y^2 = y \in \overline{R}$, $y\overline{R}(1-y) = 0$ and $(1-y)\overline{R}y = 0$, so $yx = yxy = xy$ for all $x \in \overline{R}\big)$. So $\overline{R}$ is CUC by Proposition 2.1. Assume that $x - x^2 \neq 0$ in $\overline{R}$. Since $\overline{R}$ is semipotent, there exists $\overline{0} \neq y^2 = y \in (x - x^2)\overline{R}$. So $y = (x - x^2)z$ for some $z \in \overline{R}$. Since y is central, $y = yx \cdot y(1-x) \cdot yz$, so $yx, y(1-x) \in U(y\overline{R})$. Since $yx + y(1-x) = y$, it follows from Proposition 2.1(4) that $y\overline{R}y$ is not CUC. This is a contradiction by Proposition 2.3(1). Hence $\overline{R}$ is Boolean.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6). These implications are clear.

(4) $\Rightarrow$ (5). By (4), for each $u \in U(R)$, $u - e \in J(R)$ where e is a central idempotent. Assume that $u - f \in J(R)$ for an idempotent f. Thus, $e - f \in J(R)$. Since e is central, $(e-f)^3 = e - f$. It follows that $f = e$.

(6) $\Rightarrow$ (1). Let $\overline{u} \in U(\overline{R})$. Then $\overline{u} = 0 + \overline{u}$ is clean. Assume that $\overline{u} = \overline{e} + \overline{v}$ where $\overline{e}^2 = \overline{e}$ and $\overline{v} \in U(\overline{R})$. By (6), $u = f + j_1$ and $v = g + j_2$, where $f, g \in \mathrm{idem}(R)$ and $j_1, j_2 \in J(R)$. Then $f = u - j_1 \in U(R)$, so $f = 1$. Similarly, $g = 1$. Thus, $\overline{1} = \overline{e} + \overline{1}$. So, $\overline{e} = 0$ and hence $\overline{u} \in \mathrm{ucn}(\overline{R})$. $\square$

By Lemma 2.8, *for a potent ring R, R is* UUC if and only if $R/J(R)$ is UUC; so in this case, Theorem 3.1(1) can be replaced by "R is UUC".

The ring $\mathbb{T}_2(\mathbb{Z}_2)$ is semipotent and UUC, but not CUC. Indeed, semipotent CUC rings can be characterized as follows.

Theorem 3.2. *Let R be a semipotent ring. The following are equivalent:*

(1) *R is CUC.*
(2) *R is abelian and $R/J(R)$ is Boolean.*
(3) *R is abelian and $U(R) = 1 + J(R)$.*
(4) $\mathrm{cn}(R) = \mathrm{ucn}_0(R)$.
(5) *For each $a \in \mathrm{cn}(R)$, there exists a unique $e^2 = e$ such that $a - e \in J(R)$.*

Proof. (1) $\Rightarrow$ (2). By Proposition 2.1, R is abelian. To show that $R/J(R)$ is Boolean, we repeat the argument as in proving "(1) $\Rightarrow$ (2)" of Theorem 3.1. Assume that $a-a^2 \neq 0$ in R. Since R is semipotent, there exists $0 \neq e^2 = e \in (a-a^2)R$. So $e = (a-a^2)b$ for some $b \in R$. Since e is central, $e = ea \cdot e(1-a) \cdot eb$, so $ea, e(1-a) \in U(eR)$. Since $ea + e(1-a) = e$, it follows from Proposition 2.1(4) that eRe is not CUC. This is a contradiction by Proposition 2.3(1). Hence $R/J(R)$ is Boolean.

(2) $\Leftrightarrow$ (3). The equivalences follow from Theorem 3.1.

(2) + (3) $\Rightarrow$ (4). Since $R/J(R)$ is Boolean, we have $2 \in J(R)$. Let $a = e+u$ where $e^2 = e$ and $u \in U(R)$. Then, by (3), $u = 1+j$ where $j \in J(R)$. So $a = e + (1+j) = (1-e) + (j+2e)$ where $1-e \in \mathrm{idem}(R)$ and $j+2e \in J(R)$.

(4) $\Rightarrow$ (1). By [7, Lemma 3.5], $\mathrm{ucn}_0(R) \subseteq \mathrm{ucn}(R)$. From this and (4) it follows that $\mathrm{cn}(R) = \mathrm{ucn}(R)$.

(4) $\Rightarrow$ (5). By (4), for each $a \in \mathrm{cn}(R)$, $a - e \in J(R)$ where e is a central idempotent. Assume that $a - f \in J(R)$ for an idempotent f. Then $e - f \in J(R)$. Since e is central, $(e-f)^3 = e - f$. It follows that $f = e$.

(5) $\Rightarrow$ (4). First we show that $\mathrm{nil}(R) \subseteq J(R)$. Let $a \in \mathrm{nil}(R)$. Since $a = 1 + (a-1) \in \mathrm{cn}(R)$, $a = e + j$ where $e^2 = e$ and $j \in J(R)$ by (5). So, in $R/J(R)$, $\overline{a} = \overline{e}$, implying that $\overline{e} = 0$. So $e = 0$ and hence $a \in J(R)$. Now we show that R is abelian. For $e \in \mathrm{idem}(R)$ and $x \in R$, $e = e + 0 = [e - ex(1-e)] + ex(1-e) = [e - (1-e)xe] + (1-e)xe$ where $e - ex(1-e)$, $e - (1-e)xe$ are idempotents and $ex(1-e), (1-e)xe$ are in $J(R)$. It follows from (5), $ex(1-e) = (1-e)xe = 0$, so $ex = xe$. $\square$

The assumption that R is semipotent in Theorem 3.2 is not superfluous. Indeed, $\mathbb{Z}$ is CUC. But $U(\mathbb{Z}) = \{-1, 1\}$, $1 + J(\mathbb{Z}) = \{1\}$, $\mathrm{cn}(\mathbb{Z}) = \{-1, 0, 1, 2\}$ and $\mathrm{ucn}_0(\mathbb{Z}) = \{0, 1\}$. So $U(\mathbb{Z}) \neq 1 + J(\mathbb{Z})$ and $\mathrm{cn}(\mathbb{Z}) \neq \mathrm{ucn}_0(\mathbb{Z})$.

Corollary 3.3. *A (von Neumann) regular ring is UUC if and only if it is Boolean.*

Corollary 3.4. *A ring R is uniquely clean if and only if $R = \mathrm{ucn}_0(R)$.*

A ring is uniquely clean if and only if it is clean and CUC. The following result is a bit surprising.

Theorem 3.5. *A ring is uniquely clean if and only if it is potent and CUC.*

Proof. The necessity is clear. For the sufficiency, let R be potent and CUC. It remains to show that R is clean. By Theorem 3.2, $R/J(R)$ is clean (being Boolean). As idempotents lift modulo $J(R)$, R is clean by [5, Proposition 6]. $\square$

Since uniquely clean rings are just clean CUC rings, known characterizations of uniquely clean rings in [16, Theorem 20] follow from Theorem 3.2.

By [17, Lemma 24], $R/J(R)$ is Boolean and idempotents lift modulo $J(R)$ if and only if for each $a \in R$, there exists $e^2 = e$ such that $a - e \in J(R)$; such a ring R is called *semiboolean*.

Corollary 3.6. *A ring is semiboolean if and only if it is potent and UUC.*

Proof. It follows from [17, Lemma 24] and the remark following Theorem 3.1. $\square$

A ring R is said to be *left quasi-duo* (resp., *right quasi-duo*) if every maximal left ideal (resp., maximal right ideal) of R is an ideal. Uniquely clean rings are left and right quasi-duo [16, Proposition 23]. Indeed,

Corollary 3.7. *Every potent, UUC ring is left and right quasi-duo.*

Proof. We show that $R/J(R)$ being Boolean implies that R is left and right quasi-duo. Since $R/J(R)$ is commutative, each $\overline{a}$ is central in $R/J(R)$, so $aR + J(R) = Ra + J(R)$. Let I be a maximal right ideal I of R. If $a \in R \setminus I$, $I + aR = R$, so $1 = x + ab$ where $x \in I$ and $b \in R$. Since $R/J(R)$ is Boolean, $j := a - a^2 \in J(R)$. So $a = a(x + ab) = ax + a^2b = ax + (a - j)b = ax + ab - jb = 1 + (ax - jb - x) \in 1 + (Rx + J(R)) + I \subseteq 1 + (xR + J(R)) + I = 1 + I$. We have proved that $R = I \cup (1 + I)$. It follows that I is a left ideal. $\square$

4. Group rings

In [3], the question of when a group ring is uniquely clean is addressed and the following results are obtained: If the group ring RG is uniquely clean, then R is uniquely clean and G is a 2-group; the converse holds if G is locally finite. In this section, we present some sufficient conditions for a group ring to be CUC.

The ring homomorphism $\epsilon : RG \to R$, $\Sigma r_g g \mapsto \Sigma r_g$, is called the *augmentation map*, and the kernel $ker(\epsilon)$ is called the *augmentation ideal* of RG and is denoted by $\triangle(RG)$. Note that $\triangle(RG)$ is the ideal of RG generated by $\{1 - g : g \in G\}$.

Lemma 4.1. *Suppose that every idempotent of RG is contained in R. Then RG is CUC (resp., UUC) if and only if so is R.*

Proof. The assumption indicates that RG is abelian if and only if R is abelian. Thus, we only need to verify the UUC case.

The necessity is by Proposition 2.3. For the sufficiency, assume that $u = v + e$ where u, v are units in RG and $e^2 = e \in RG$. Then $\epsilon(u) = \epsilon(v) + e$, where $\epsilon(u), \epsilon(v) \in U(R)$ and $e \in \text{idem}(R)$. Since R is UUC, it follows that $e = 0$. So RG is UUC. $\square$

Proposition 4.2. *The integral group ring of an arbitrary group G is CUC.*

Proof. By [18, Corollary 7.2.4], $\mathbb{Z}G$ contains only trivial idempotents. Since $\mathbb{Z}$ is CUC, $\mathbb{Z}G$ is CUC by Lemma 4.1. $\square$

Proposition 4.3. *Let R be a ring and G be a torsion-free abelian group. Then RG is CUC if and only if R is CUC.*

Proof. We only need to show the sufficiency. Assume that R is CUC. To show that RG is CUC, we can assume that G is finitely generated. Then there exists $n \geq 1$ such that $G \cong G_1 \times G_2 \times \cdots \times G_n$ where $G_i \cong C_\infty$, the infinite cyclic group, for $i = 1, \ldots, n$. Since $RG \cong (\cdots((RG_1)G_2)\cdots)G_n$, it suffices to show that RC_∞ is CUC. By Proposition 2.1, R is abelian, so all idempotents of $R[t, t^{-1}]$

are contained in R by [6, Theorem 5]. Since $R[t,t^{-1}]\cong RC_\infty$, we deduce that the idempotents of RC_∞ are contained in R. It follows from Lemma 4.1 that RC_∞ is CUC. $\square$

Proposition 4.4. *Let G be a locally finite group.*

(1) *If R is a potent ring, then RG is UUC if and only if R is UUC and G is a 2-group.*

(2) *If R is a semipotent ring, then RG is CUC if and only if R is CUC and G is a 2-group.*

Proof. (1)($\Rightarrow$). By Proposition 2.3, R is UUC. Assume that G is not a 2-group. Then G contains an element g of prime order $p>2$. As a subring of RG, $R\langle g\rangle$ is UUC. Since R is potent and UUC, $R/J(R)$ is Boolean, so $p\in U(R)$. Since $t^{p-1}+t^{p-2}+\cdots t+1=(t-1)[t^{p-2}+2t^{p-3}+\cdots+(p-2)t+(p-1)]+p$, it follows that $(t-1)+(t^{p-1}+t^{p-2}+\cdots+t+1)=R[t]$ and hence, $R\langle g\rangle\cong R[t]/(t^p-1)\cong R[t]/(t-1)\oplus R[t]/(t^{p-1}+t^{p-2}+\cdots+t+1)$. By Proposition 2.3, $S:=R[t]/(t^{p-1}+t^{p-2}+\cdots+t+1)$ is UUC. In S, $(t^2+t)[t^{p-3}+t^{p-5}+\cdots+t^2+1]=t^{p-1}+t^{p-2}+\cdots+t=-1\in U(S)$, so $t,t+1\in U(S)$. But $(t+1)+(-t)=1$, contradicting that S is UUC. So G is a 2-group.

(1)($\Leftarrow$). Assume that $u=v+e$ where $u,v\in U(RG)$ and $e=\sum e_g g\in \text{idem}(RG)$. Then $\epsilon(u)=\epsilon(v)+\epsilon(e)$, where $\epsilon(u),\epsilon(v)\in U(R)$ and $\epsilon(e)\in\text{idem}(R)$. Since R is UUC, $\epsilon(e)=0$, so $e=\sum e_g g-\sum e_g=\sum e_g(g-1)\in\triangle(RG)$. But, by [21, Lemma 2], $\triangle(RG)\subseteq J(RG)$, so $e\in J(RG)$. It follows that $e=0$. So RG is UUC.

(2)($\Rightarrow$). By Proposition 2.3, R is CUC. Since R is semipotent, $R/J(R)$ is Boolean by Theorem 3.2. The proof of "(1)($\Rightarrow$)" clearly indicates that R CUC and $R/J(R)$ Boolean implies that G is a 2-group.

(2)($\Leftarrow$). Since R is semipotent and CUC, R is abelian and $R/J(R)$ is Boolean by Theorem 3.2. So $2\in J(R)$. By [3, Lemma 11], the idempotents of RG are contained in R. Hence, by Lemma 4.1, RG is CUC. $\square$

5. All factors are CUC

Factor rings of uniquely clean rings are uniquely clean [16]. But factor rings of CUC rings need not be UUC: $\mathbb{Z}$ is CUC but $\mathbb{Z}_{p^k}$ is not UUC for any prime $p > 2$ and any integer $k \geq 1$. The ring R in the next example is a CUC ring but idempotents do not lift modulo $J(R)$ and $R/J(R)$ is not UUC.

Example 5.1. *Let* $R = \{\frac{m}{n} \in \mathbb{Q} : \gcd(n, 6) = 1\, (\frac{m}{n}$ *in lowest term*$)\}$. *One easily sees that* 1 *is not* 2*-good in* R*, so* R *is CUC by Examples 2.2(1). It is known that* $J(R) = 2R \cap 3R$ *and idempotents do not lift modulo* $J(R)$ *(see* [2, Exercise 4, page 312]*). Moreover,* $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$*, which is not UUC (since* $\mathbb{Z}_3$ *is not UUC).*

It is noteworthy to characterize the ring whose every factor is CUC (resp., UUC). This question has an answer for commutative rings and semipotent rings.

Proposition 5.2. *If* $R/J(R)$ *is Boolean, then every factor of* R *is UUC.*

Proof. For any ideal I of R, write $J(R/I) = K/I$ where K is an ideal of R with $J(R) \subseteq K$. So $(R/I)/J(R/I) \cong R/K$ is an image of $R/J(R)$. It follows that $(R/I)/J(R/I)$ is Boolean, and hence is UUC. So R/I is UUC by Remark 2.10. $\square$

Proposition 5.3. *Let* R *be a commutative ring. The following are equivalent:*

(1) *Every factor of* R *is CUC.*
(2) *Every factor of* R *is UUC.*
(3) $R/J(R)$ *is Boolean.*

Proof. (1) $\Rightarrow$ (3). A commutative primitive CUC ring is a field that is CUC, so it is $\mathbb{Z}_2$ by Corollary 2.6. Hence, $R/J(R)$ is a subdirect product of $\mathbb{Z}_2$'s, so it is Boolean.

(3) $\Rightarrow$ (2). This is by Proposition 5.2.

(2) $\Rightarrow$ (1). Since R is commutative, every factor of R is commutative and the implication follows from Proposition 2.1. $\square$

Proposition 5.4. *Let R be a semipotent ring.*

(1) *Every factor of R is UUC if and only if $R/J(R)$ is Boolean.*

(2) *Every factor of R is CUC if and only if $ax - xa \in R(a - a^2)R$ for all $a, x \in R$ and $R/J(R)$ is Boolean.*

Proof. (1) The sufficiency is by Proposition 5.2. For the necessity, $R/J(R)$ is UUC (being a factor of R) and hence is Boolean by Theorem 3.1.

(2) Note that every factor of R is abelian if and only if $ax - xa \in R(a - a^2)R$ for all $a, x \in R$. Indeed, to see the necessity, for $a \in R$, $\overline{a}$ is an idempotent in $R/R(a - a^2)R$, so is central. Hence, $ax - xa \in R(a - a^2)R$ for all $x \in R$. To see the sufficiency, if $a + I$ is an idempotent of R/I where I is an ideal I of R, then $a^2 - a \in I$, so $R(a - a^2)R \subseteq I$. Thus, $ax - xa \in I$ for all $x \in R$, so $a + I$ is central in R/I. Hence R/I is abelian. Now (2) follows from the above note, (1) and Proposition 2.1. $\square$

We conclude with

Question 5.5. *Is the converse of Proposition 5.2 true?*

Acknowledgments

The authors are very grateful to the referee for the careful reading of the manuscript and for his/her valuable comments and suggestions that have largely improved the presentation. Zhou was supported by a Discovery Grant (RGPIN-2016-04706) from NSERC of Canada.

References

[1] D. D. Anderson and V. P. Camillo, Commutative rings whose elements are a sum of a unit and idempotent. *Comm. Algebra* **30** (2002), 3327-3336.

[2] F.W. Anderson and K.R. Fuller, Rings and Categories of Modules, Springer-Verlag, 1974.

[3] J. Chen, W.K. Nicholson and Y. Zhou, Group rings in which every element is uniquely the sum of a unit and an idempotent. *J. Algebra* **306** (2006), 453-460.

[4] F.J. Hall, R.E. Hartwig, I.J. Katz and M. Newman, Pseudo-similarity and partial unit regularity. *Czech. Math. J.* **33**(3) (1983), 361-372.

[5] J. Han and W.K. Nicholson, Extensions of clean rings. *Comm. Algebra* **29**(6) (2001), 2589-2595.

[6] P. Kanwar, A.Leroy and J. Matczuk, Idempotents in ring extensions. *J. Algebra* **389**(2) (2013), 128-136.

[7] D. Khurana, T.Y. Lam, P.P. Nielsen and Y. Zhou, Uniquely clean elements in rings. *Comm. Algebra* **43**(5) (2015), 1742-1751.

[8] T. Koşan, S. Sahinkaya and Y. Zhou, On weakly clean rings. *Comm. Algebra* **45**(8) (2017), 3494-3502.

[9] T. Koşan, T-K. Lee and Y. Zhou, An extension of rings and Hochschild 2-cocycles. *Contemporary ring theory 2011*, 29-46, World Sci. Publ., Hackensack, NJ, 2012.

[10] T-K. Lee and Y. Zhou, Clean index of rings. *Comm. Algebra* **40**(3) (2012), 807-822.

[11] T-K. Lee and Y. Zhou, Rings of clean index 4 and applications. *Comm. Algebra* **41**(1) (2013), 238-259.

[12] T-K. Lee and Y. Zhou, From Boolean rings to clean rings. *Ring theory and its applications*, 223â€“232, Contemp. Math., 609, Amer. Math. Soc., Providence, RI, 2014.

[13] J. Levitzki, On the structure of algebraic algebras and related rings. *Trans. AMS.* **74**(1953), 384-409.

[14] W.K. Nicholson, I-rings. *Trans. AM.* **207** (1975), 361-373.

[15] W.K. Nicholson and M.F. Yousif, Quasi-Frobenius rings. Cambridge Tracts in Mathematics, **158**. *Cambridge University Press*, Cambridge, 2003.

[16] W.K. Nicholson and Y. Zhou, Rings in which elements are uniquely the sum of an idempotent and a unit. *Glasg. Math. J.* **46**(2) (2004), 227-236.

[17] W.K. Nicholson and Y. Zhou, Clean general rings. *J. Algebra* **291**(1) (2005), 297-311.

[18] C. Polcino Milies and S.K. Sehgal, An introduction to group rings. Kluwer Academic Publishers, Dordrecht, 2002.

[19] A. Tuganbaev, Rings close to regular. Kluwer Academic Publishers, Dordrecht, 2002.

[20] C. Wu and L. Zhao, RS rings and their applications. *Publ. Math. Debrecen* **96** (1-2) (2020), 77-90.

[21] Y. Zhou, On clean group rings. *Advances in ring theory*, 335-345, Trends Math., Birkhäuser/Springer, Basel, 2010.

Department of Mathematics, Faculty of Mathematics and Computer Science, Babes-Bolyai University, Ro-400084 Cluj-Napoca, Romania

E-mail address: `calu@math.ubbcluj.ro`

Department of Mathematics and Statistics, Memorial University of Newfoundland, St.John's, NL A1C 5S7, Canada

E-mail address: `zhou@mun.ca`