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A 3×3 NILPOTENT MATRIX OF INDEX 3 WHICH HAS UNIT STABLE RANGE ONE

GRIGORE CĂLUGĂREANU AND HORIA F. POP

Abstract. The main goal of this paper is to show which are the problems we face when trying to check that a 3×3 nilpotent matrix has (unit) stable range one. Actually we focus on the 3×3 matrix with 2 on the superdiagonal and zeros elsewhere.

We first show that over Bézout domains nilpotent 2×2 and 3×3 matrices of index 2, have (unit) stable range one. Then, preparing the proof in the last section, over any commutative elementary divisor ring, we characterize some completions of matrices to invertible matrices by using their diagonal reductions. Finally, using these, we prove the statement in the title.

1. Introduction

All rings we consider are associative with identity. For a ring R, we denote by $U(R)$, $N(R)$ and $J(R)$ the set of all units of a ring R, the set of all nilpotents of R and the Jacobson radical of R, respectively. By E_{ij} we denote the $n \times n$ matrix having all entries equal to zero, excepting the (i,j) entry which is 1. For a square matrix T over any commutative ring, $\mathrm{Tr}(T)$ denotes the *trace* of T and $\det(T)$ denotes the *determinant* of T.

An element a of a ring R has *left stable range one* (*sr1*, for short) if for any $x \in R$ satisfying $Ra + Rx = R$, there exists $y \in R$ such that $a + yx$ is a unit. Equivalently, a has left sr1 if for every $x \in R$ there exists $y \in R$ such that $a + y(xa - 1)$ is a unit. If we can choose $y \in U(R)$ then a has *unit sr1*. Symmetrically, (unit) right stable range one elements are defined. A ring has left (or right) stable range one if all its elements have left (or right) sr1. It is known that the sr1 condition is left-right symmetric *for rings* but may not be left-right symmetric *for elements* of a ring. Therefore in the sequel (and specifically for matrices) we refer to the left sr1 condition.

Since units, idempotents and elements in the Jacobson radical have stable range one in any ring, it is natural to ask whether there is a *nilpotent element of a ring, which has not stable range one*. To the best of our knowledge such an example was not found so far.

When writing this paper, our initial goal was to find such an example, and, as customarily, one starts by searching among 2×2 or 3×3 matrices over as general as possible (commutative) rings. Notice that, if R is an *exchange* ring (i.e., for every $x \in R$ there exists an idempotent $e \in R$ such that $e \in Rx$ and $1 - e \in (1 - x)R$, a very large class of rings) and $N(R)$ is a subring of R then $N(R) \subseteq J(R)$, so nilpotents have sr1.

To check that some given matrix has (unit) sr1 is a difficult task.

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For 2×2 matrices, sr1 can be characterized (and checked) using some quadratic Diophantine like equations (see [3]) which for integral matrices can be solved using suitable (existing on Internet) software.

To check whether a 3×3 matrix has (unit) sr1, over some integral domain or even over the integers, is harder.

In section 2 we give some general results on $n\times n$ zero-square matrices and in section 3 we show that over Bézout domains, 2×2 and 3×3 zero-square matrices are similar to multiples of E_{12} or E_{13}, respectively, so have (unit) sr1, since any multiple $rE_{ij}\in \mathbb{M}_n(R)$ has it (over any ring with identity). Therefore, an example of nilpotent that has not (unit) sr1 does not exist in $\mathbb{M}_2(\mathbb{Z})$ and does not exist in $\mathbb{M}_3(\mathbb{Z})$ for index 2 nilpotents. Hence, in searching for such an example in $\mathbb{M}_3(\mathbb{Z})$, we should consider index 3 nilpotents.

In the last section we focus on the nilpotent matrix of index 3, $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. It turned out that to prove this simple nilpotent 3×3 matrix *has* unit sr1, is not easy and we had to prepare (in section 4) results on some specific completions of arbitrary $n\times n$ matrices to invertible $(n+1)\times(n+1)$ matrices over commutative (Henriksen) elementary divisor rings. In the sequel, the word "completion" will be used only in this sense.

This way, we changed the initial goal and all the results in our paper first motivate the choice of this nilpotent matrix and finally contribute to prove, in the last section, that this nilpotent matrix *has unit stable range* 1.

Therefore, finding a nilpotent 3×3 matrix (if any) which has not sr1, remains an open problem.

2. Zero-square $n\times n$ matrices

For a zero-square $n\times n$ matrix T over a commutative (unital) ring, denote by T_{ab}^{cd} the 2×2 minor on the rows a and b and on the columns c and d. A simple computation of $\mathrm{row}_i(T)\cdot \mathrm{col}_j(T)$ for $i\neq j$ or $i=j$ gives

Proposition 1. *Let $T=[t_{ij}]_{1\le i,j\le n}$ be an $n\times n$ matrix over a commutative ring R and let $t_{ij}^{(2)}$ be the entries of T^2. Then*

$$\begin{array}{lcll} t_{ij}^{(2)} & = & t_{ij}\mathrm{Tr}(T)+\sum_{k\in\{1,\dots,n\}-\{i,j\}} T_{ik}^{kj} & i\neq j \\ t_{ii}^{(2)} & = & t_{ii}\mathrm{Tr}(T)+T_{i1}^{1i}+\dots+T_{i,i-1}^{i-1,i}+T_{i,i+1}^{i+1,i}+\dots+T_{in}^{ni} & i=j \end{array}.$$

First recall that the *rank* of a (not necessarily square) matrix A (denoted $\mathrm{rk}(A)$) can be defined *over any commutative ring* R, using the annihilators of the ideals $I_t(A)$ generated by the $t\times t$ minors of A (see e. g. [1]). In particular, $\mathrm{rk}(A)=1$ if all 2×2 minors are zero and these two conditions are equivalent over integral domains. Then it can be shown that *equivalent matrices* (in particular, similar matrices) *have the same rank* (see [1], **4.11**).

Therefore

Corollary 2. *Let T be an $n\times n$ matrix over any commutative ring. If all 2×2 minors of T are zero and $\mathrm{Tr}(T)=0$ then $T^2=0_n$.*

Remark. Over integral domains a (well-known) converse also holds: *If $T^2=0_n$ then* $\det(T)=\mathrm{Tr}(T)=0$.

Over integral domains, in order to have a characterization of form

$$T^2 = 0_n \text{ if and only if } \mathrm{rk}(T) = 1 \text{ and } \mathrm{Tr}(T) = 0,$$

the only remaining implication is that $T^2 = 0_n$ and $\mathrm{Tr}(T) = 0$ imply $\mathrm{rk}(T) = 1$ (i.e. all 2×2 minors of T equal zero).

In what follows we show that this implication holds over any commutative ring for $n = 3$ if 2 is *not* a zero divisor, but fails for any $n \geq 4$.

Theorem 3. *Let R be a commutative ring such that 2 is not a zero divisor and let $T \in \mathbb{M}_3(R)$ with $\mathrm{Tr}(T) = 0$. Then $T^2 = 0_3$ if and only if all 2×2 minors of T equal zero.*

Proof. To avoid too many indexes and emphasize the diagonal elements (i.e. the zero trace) we write $T = \begin{bmatrix} x & a & c \\ b & y & e \\ d & f & -x-y \end{bmatrix}$.

If $\mathrm{Tr}(T) = 0$, the condition $T^2 = 0_3$ is *equivalent* to the following nine LHS equalities

$$\begin{array}{rll} x^2 + ab + cd = 0 & (1) & \\ a(x+y) + cf = 0 & (2) & T_{13}^{23} = 0 \\ ae = cy & (3) & T_{12}^{23} = 0 \\ b(x+y) + de = 0 & (4) & T_{23}^{13} = 0 \\ y^2 + ab + ef = 0 & (5) & \\ bc = ex & (6) & T_{12}^{13} = 0 \\ bf = dy & (7) & T_{23}^{12} = 0 \\ ad = fx & (8) & T_{13}^{12} = 0 \\ (x+y)^2 + cd + ef = 0 & (9) & \end{array}.$$

The two terms equalities (i.e., (3), (6), (7), (8)) are *equivalent* to the vanishing of four 2×2 minors (see the RHS column of zero minors). Further, two other equalities, namely, (2) and (4), are *equivalent* to the vanishing of another two minors.

Thus, this equivalently covers the six *off diagonal* 2×2 minors. What remains are the vanishing of the three 2×2 *diagonal* minors.

From $x^2 + ab + cd = 0$, $y^2 + ab + ef = 0$ and $(x+y)^2 + cd + ef = 0$ we get (since 2 is not a zero divisor) $xy = ab$, and so another zero 2×2 minor. Finally using $x^2 + ab + cd = 0$, $y^2 + ab + ef = 0$ and $xy = ab$, we get the last two zero 2×2 diagonal minors: $x(x+y) + cd = 0$ and $y(x+y) + ef = 0$.

The converse was settled in the general $n \times n$ case (see Corollary 2). □

Remark. The hypothesis "2 *is not a zero divisor*" is essential for the vanishing of the three diagonal 2×2 minors (over any commutative ring). Consider $R = \mathbb{Z}_2[X, Y]/I$ for $I := (X^2, Y^2)$ and the diagonal matrix over R, $T = \begin{bmatrix} X+I & 0 & 0 \\ 0 & Y+I & 0 \\ 0 & 0 & X+Y+I \end{bmatrix}$. Then $T^2 = 0_3$, $\mathrm{Tr}(T) = 0$, but the diagonal minors are not zero. Clearly, 2 is a zero divisor in R.

Before dealing with the 3×3 matrices case, here is an example of 4×4 zero-square matrix (over any commutative ring) with zero trace and rank 2.

Example. $C_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}^2 = 0_4$, has zero trace but many not zero 2×2 minors (e.g. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, in the center).

Hence $T^2 = 0_4$ does not generally imply $\text{rk}(T) = 1$. Adding to this example as many zero rows and zero columns as necessary, we see that $T^2 = 0_n$ does not generally imply $\text{rk}(T) = 1$, for any $n \geq 5$.

Since nonzero multiples of E_{1n} have rank 1, and similar matrices have the same rank, we obtain

Theorem 4. *Over any commutative ring and for every $n \geq 4$, there are $n \times n$ zero-square matrices which are not similar to any multiple of E_{1n}.*

3. The zero-square 3×3 case

A ring R is called a *GCD* ring if every pair a, b of nonzero elements has a greatest common divisor, denoted by $\gcd(a, b)$. A GCD ring R is *Bézout* if whenever $\delta = \gcd(a, b)$, there exist $s, t \in R$ such that $sa + tb = \delta$. If $\delta = 1$, the elements a, b are called *coprime*.

Notice that since our main results are proved over Bézout rings, in the sequel equalities are written (as customarily) modulo association (in divisibility). A row $\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$ of elements in a ring R is called *unimodular* if $a_1R + \ldots + a_nR = R$. When convenient, a unimodular row will be identified with $(a_1, ..., a_n) \in R^n$.

We just mention that over Bézout domains, any zero-square 2×2 matrix is similar to a multiple of E_{12} (for a proof, see Proposition 4.3, [4]). Hence it has unit sr1, since, more generally, any multiple $rE_{ij} \in \mathbb{M}_n(R)$ has unit sr1 (see [4]). This section is devoted to prove an analogous result for 3×3 matrices over Bézout domains.

Before proving the main result of this section, we prove a useful lemma and proposition.

Lemma 5. *Let a, b, c, a', b', c' be elements in a GCD domain R. If $ab' = a'b$, $ac' = a'c$, $bc' = b'c$ and the rows $\begin{bmatrix} a & b & c \end{bmatrix}$ and $\begin{bmatrix} a' & b' & c' \end{bmatrix}$ are unimodular then the pairs a, a', b, b' and c, c' are (respectively) associated (in divisibility). Moreover, there exists a unit $u \in U(R)$ such that $\begin{bmatrix} a' & b' & c' \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} u$.*

Proof. Denote $\delta = \gcd(a, b)$ with $a = \delta a_1$, $b = \delta b_1$ and $\delta' = \gcd(a', b')$ and $a' = \delta' a'_1$, $b' = \delta' b'_1$. From $ab' = a'b$ cancelling $\delta\delta'$ we obtain $a_1 b'_1 = a'_1 b_1$. Since a_1, b_1 are coprime, it follows $a_1 \mid a'_1$. Symmetrically, since a'_1, b'_1 are coprime, it follows $a'_1 \mid a_1$, so that a_1, a'_1 are associates. Hence there is a unit $u \in U(R)$ such that $a_1 = a'_1 u$.

Further, notice that $\gcd(\delta, c) = \gcd(\gcd(a, b), c) = 1$ and so δ, c are coprime. Now we use $ac' = a'c$, that is, $\delta(a'_1 u)c' = \delta a_1 c' = \delta' a'_1 c$. Cancelling a'_1 we get $\delta u c' = \delta' c$ and since δ, c are coprime, $\delta \mid \delta'$. Symmetrically, $\delta' \mid \delta$ and so δ, δ' are also associates. Therefore $a = \delta a_1$ and $a' = \delta' a'_1$ are associates.

In a similar way, it follows that b, b' and c, c' are associates, respectively.

Finally, suppose $a' = au$, $b' = bv$ and $c' = cw$ for some $u, v, w \in U(R)$. From $ab' = a'b$ we get $abv = aub$, so $v = u$. Analogously, $w = v$ and so $w = v = u$, as claimed. □

Remark. We can state as $\mathrm{rk}\begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix} = 1$, the second hypothesis of this lemma.

Proposition 6. *Let R be a GCD domain and let a, b, c, a', b', c' be elements of R. If* $\mathrm{rk}\begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix} = 1$ *(i.e., $ab' = a'b$, $ac' = a'c$, $bc' = b'c$), $\delta = \gcd(a, b, c)$, $\lambda = \gcd(a', b', c')$ and $a = \delta a_1$, $b = \delta b_1$, $c = \delta c_1$, $a' = \lambda a'_1$, $b' = \lambda b'_1$, $c' = \lambda c'_1$, then a_1, b_1, c_1 and a'_1, b'_1, c'_1 are (respectively) associated (in divisibility). Moreover, $\begin{bmatrix} a'_1 & b'_1 & c'_1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} u$ for some $u \in U(R)$.*

Proof. We just use the previous lemma. □

In the sequel, for 3-vectors we use the well-known operations of dot product, cross product and scalar triple product.

Definition. The 3-vector $\mathbf{a} = (a_1, a_2, a_3) \in R^3$ is *unimodular* iff the ideal generated by its components is the whole ring, i.e. $I = (a_1, a_2, a_3) = Ra_1 + Ra_2 + Ra_3 = R$. Equivalently, there exists $\mathbf{b} = (b_1, b_2, b_3) \in R^3$ such that $\mathbf{a} \cdot \mathbf{b} = 1$.

More detailed, for 3 elements of a ring $a_1, a_2, a_3 \in R$, the ideal generated by these $I = (a_1, a_2, a_3) = Ra_1 + Ra_2 + Ra_3$ can be the whole ring R, case when $\{a_1, a_2, a_3\}$ (ideal) generates R, or else, it is *not* the whole ring. Since by Zorn's Lemma, every proper ideal is included in a maximal ideal, the second case can be characterized as follows: the system $\{a_1, a_2, a_3\}$ is *not* an (ideal) generating system if and only if these elements (and so is the ideal these generate) are included in a maximal ideal.

This way, a 3-vector $\mathbf{a} = (a_1, a_2, a_3) \in R^3$ is unimodular if and only if $\{a_1, a_2, a_3\}$ is not included in any maximal ideal M of R. Equivalently, for every maximal ideal M of R, at least one of the $a_i \notin M$, or else, at least one of $a_1 + M, a_2 + M, a_3 + M \in R/M$ is $\neq M$ (i.e. is not zero in R/M).

To simplify the writing, we denote $\mathbf{a} + M = (a_1 + M, a_2 + M, a_3 + M)$, which can be viewed as a 3-vector in $(R/M)^3$. Moreover, we extend accordingly the dot product $(\mathbf{a} + M) \cdot (\mathbf{b} + M) = \mathbf{a} \cdot \mathbf{b} + M \in R/M$.

Proposition 7. *Suppose $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$ are unimodular 3-vectors such that $\mathbf{a} \cdot \mathbf{b} = 1$, $\mathbf{a} \cdot \mathbf{c} = 0$. Then the cross product $\mathbf{b} \times \mathbf{c}$ is also a unimodular row.*

Proof. As mentioned above, it suffices to show that the 3-vector $\mathbf{b} \times \mathbf{c}$ (as customarily identified with the (ideal) generating system $\{b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1\}$) is nonzero, modulo any maximal ideal. Since R is a commutative (unital) ring, modulo any maximal ideal M of R, R/M is a field and (with the above notation) $\mathbf{b} + M$, $\mathbf{c} + M$ are nonzero 3-vectors in $(R/M)^3$ (otherwise these are not unimodular). It is easy to see that these two vectors are linearly independent (indeed, if $(\mathbf{a} + M) \cdot (\mathbf{b} + M) = 1 + M$, $(\mathbf{a} + M) \cdot (\mathbf{c} + M) = M$ and $\mathbf{b} + M = k(\mathbf{c} + M)$ for some $k \in R$, then $1 + M = M$, impossible). Hence their cross product is (well-known to be) nonzero and the proof is complete. □

Proposition 8. *Let R be a commutative ring and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be unimodular 3-vectors such that $\mathbf{a} \cdot \mathbf{b} = 1$ and $\mathbf{a} \cdot \mathbf{c} = 0$. There exists a unimodular 3-vector $\mathbf{x}$, also orthogonal on $\mathbf{a}$, such that $\mathbf{b} \cdot (\mathbf{x} \times \mathbf{c}) = 1$.*

Proof. By the above proposition, since $\mathbf{b} \times \mathbf{c}$ (which is just the three 2×2 minors of the matrix $[\mathbf{bc}] = \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$) is also unimodular, there exists a unimodular 3-vector $\mathbf{x}$ such that $(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{x} = 1$, that is, $\det[\mathbf{bcx}] = 1$.

Hence $\mathbf{b} \cdot (\mathbf{x} \times \mathbf{c}) = 1$. If $\mathbf{a} \cdot \mathbf{x} = s$, replace $\mathbf{x}$ by $\mathbf{x} - s\mathbf{b}$ and then this vector is also orthogonal on $\mathbf{a}$ (indeed, $\mathbf{a} \cdot (\mathbf{x} - s\mathbf{b}) = \mathbf{a} \cdot \mathbf{x} - s(\mathbf{a} \cdot \mathbf{b}) = s - s = 0$). □

We are now ready to prove our main result

Theorem 9. *Zero-square* 3×3 *matrices over any Bézout domain, are similar to multiples of* E_{13}.

Proof. Again consider $T = \begin{bmatrix} x & a & c \\ b & y & e \\ d & f & -x-y \end{bmatrix}$ with $T^2 = 0_3$ (by Theorem 3, $\mathrm{rank}(T) = 1$, that is, all 2×2 minors are zero).

Denote $\delta = \gcd(x, a, c)$, $\lambda = \gcd(b, y, e)$ and $\gamma = \gcd(d, f, x+y)$ so that $x = \delta x_1$, $a = \delta a_1$, $c = \delta c_1$, $b = \lambda b_1$, $y = \lambda y_1$, $e = \lambda e_1$, $d = \gamma d_1$, $f = \gamma f_1$ and $x+y = \gamma(x_2+y_2)$.

According to Proposition 6, there are units u, v such that $\begin{bmatrix} b_1 & y_1 & e_1 \end{bmatrix} = \begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} u$ and $\begin{bmatrix} d_1 & f_1 & -x_2 - y_2 \end{bmatrix} = \begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} v$.

Hence $T = \begin{bmatrix} \delta x_1 & \delta a_1 & \delta c_1 \\ \lambda u x_1 & \lambda u a_1 & \lambda u c_1 \\ \gamma v x_1 & \gamma v a_1 & \gamma v c_1 \end{bmatrix}$ and since $\begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix}$ is unimodular, there are $s, t, z \in R$ and $s x_1 + t a_1 + z c_1 = 1$.

Note that $\mathrm{Tr}(T) = \delta x_1 + \lambda u a_1 + \gamma v c_1 = 0$.

Denote $r = \gcd(\delta, \lambda, \gamma) = \gcd(T)$ and $\delta = r\delta_1$, $\lambda = r\lambda_1$, $\gamma = r\gamma_1$. We are looking for an invertible matrix U such that $TU = U(rE_{13}) = \begin{bmatrix} 0 & 0 & ru_{11} \\ 0 & 0 & ru_{21} \\ 0 & 0 & ru_{31} \end{bmatrix}$.

Our choice for r is necessary: indeed, writing $T = rUE_{13}U^{-1}$, we see that r must divide all the entries of T. Also note that, if $\det(U) = 1$, every row and every column of U must be unimodular.

We choose $\mathrm{col}_3(U) = \begin{bmatrix} s \\ t \\ z \end{bmatrix}$. By computation

$ru_{11} = \mathrm{row}_1(T) \cdot \mathrm{col}_3(U) = \delta(x_1 u_{13} + a_1 u_{23} + c_1 u_{33}) = \delta$,
$ru_{21} = \mathrm{row}_2(T) \cdot \mathrm{col}_3(U) = \lambda u(x_1 u_{13} + a_1 u_{23} + c_1 u_{33}) = \lambda u$,
$ru_{31} = \mathrm{row}_3(T) \cdot \mathrm{col}_3(U) = \gamma v(x_1 u_{13} + a_1 u_{23} + c_1 u_{33}) = \gamma v$, and,
$\begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} = 0$ and so
$\begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} U = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

Hence, the first column of U must be $\mathrm{col}_1(U) = \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} \frac{\delta}{r} \\ \frac{\lambda}{r} u \\ \frac{\gamma}{r} v \end{bmatrix}$. These fractions exist since $r = \gcd(\delta, \lambda, \gamma)$.

We indeed have $\begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} \begin{bmatrix} \frac{\delta}{r} \\ \frac{\lambda}{r}u \\ \frac{\gamma}{r}v \end{bmatrix} = \frac{1}{r}(\delta x_1 + \lambda u a_1 + \gamma v c_1) = \frac{1}{r}(x + y - (x + y)) = 0$ (because $\mathrm{Tr}(T) = 0$).

Finally, we need a suitable column $\mathrm{col}_2(U)$ such that $U = \begin{bmatrix} \delta_1 & u_{12} & s \\ \lambda_1 u & u_{22} & t \\ \gamma_1 v & u_{32} & z \end{bmatrix}$ is invertible and $\begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} U = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

Taking $\mathbf{a} = [x_1\ a_1\ c_1]$, $\mathbf{b} = [s\ t\ z]$ and $\mathbf{c} = [\delta_1\ \lambda_1 u\ \gamma_1 v]$ the existence of $\mathbf{x} = [u_{12}\ u_{22}\ u_{32}]$ follows (by transpose) from the previous proposition. □

Example. Take $x_1 = 6$, $a_1 = 10$, $c_1 = 15$, so that no two of these are coprime and $T = \begin{bmatrix} -180 & -300 & -450 \\ 90 & 150 & 225 \\ 12 & 20 & 30 \end{bmatrix}$, $\delta = -30$, $\lambda = 15$, $\gamma = 2$ and so $r = 1$.

In order to find the third column of U, we first we solve the linear Diophantine equation, $6s + 10t + 15z = 1$. We denote $w = 3s + 5t$ and solve $2w + 15z = 1$. This gives $w = -7 + 15n$, $z = 1 - 2n$.

We choose $w = -7$ (for $n = 0$) and solve $3s + 5t = -7$. This gives for instance $s = -14$, $t = 7$, so we choose also $z = 1$ and $U = \begin{bmatrix} -30 & u_{12} & -14 \\ 15 & u_{22} & 7 \\ 2 & u_{32} & 1 \end{bmatrix}$.

As for the second column of U, we have the equation

$$-u_{12} \det \begin{bmatrix} 15 & 7 \\ 2 & 1 \end{bmatrix} + u_{22} \det \begin{bmatrix} -30 & -14 \\ 2 & 1 \end{bmatrix} - u_{32} \det \begin{bmatrix} -30 & -14 \\ 15 & 7 \end{bmatrix} = 1,$$

that is, $-u_{12} - 2u_{22} = 1$.

Hence $2u_{22} = -1 - u_{12}$ and so $6u_{12} - 5 - 5u_{12} + 15u_{32} = 0$ or $u_{12} + 15u_{32} = 5$. We can choose $u_{12} = 5$, $u_{32} = 0$ and so $u_{22} = -3$.

Indeed $TU = \begin{bmatrix} -180 & -300 & -450 \\ 90 & 150 & 225 \\ 12 & 20 & 30 \end{bmatrix} \begin{bmatrix} -30 & 5 & -14 \\ 15 & -3 & 7 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -30 \\ 0 & 0 & 15 \\ 0 & 0 & 2 \end{bmatrix}$ $= UE_{13}$, so T is similar to E_{13}.

4. Completions over elementary divisor rings

The rings we consider in this section are commutative with identity. We use the terminology from [6].

Definition. A ring R is called a (*Henriksen*) *elementary divisor* ring if for every $n \times n$ matrix A there exist invertible $n \times n$ matrices P, Q such that PAQ is a diagonal matrix (called its *diagonal reduction*). Briefly, in such rings every $n \times n$ matrix is equivalent to a diagonal matrix.

A ring is called a *Hermite* ring if every square matrix admits a *triangular reduction* (i.e. is equivalent to an upper triangular matrix). Thus (Henriksen) elementary divisor rings are Hermite and Hermite rings are Bézout.

Following [7], *we just mention* (but not use) that among these, a ring is called (*classical*) *elementary divisor* ring (in the sense of Kaplansky) if in the diagonal matrix PAQ each element divides the element below. Such reductions are called *canonical* diagonal reductions. Principal ideal rings are (classical and so also Henriksen) elementary divisor rings, and unit-regular rings, semichain rings, separative (Von Neumann) regular rings are (Henriksen) elementary divisor rings.

For an $n \times n$ matrix $A = [a_{ij}]_{1 \leq i,j \leq n}$ we use the notation $\gcd(A) = \gcd\{a_{ij} : 1 \leq i, j \leq n\}$. If $\gcd(A) = 1$ we say that the entries of A are (collectively) *coprime*.

We use the block notation: if $A = [a_{ij}]_{1 \leq i,j \leq n}$ is an $n \times n$ matrix over a ring R then $U = \begin{bmatrix} A & \alpha \\ \beta & t \end{bmatrix}$ is its completion to an $(n+1) \times (n+1)$ matrix, with an n-column α, an n-row β and $t \in R$. The matrix A is said to be *completable* if it has an *invertible* completion.

Before proving our characterizations, some prerequisites are gathered in the following

Lemma 10. *(i) A is completable iff the transpose A^T is completable.*

(ii) If A is completable and B is equivalent to A, then B is also completable.

(iii) Let B be equivalent to A. Then $\gcd(A) = 1$iff $\gcd(B) = 1$.

(iv) A is completable only if its entries are (collectively) coprime.

Proof. (i) It suffices to transpose the completion U.

(ii) For two $n \times n$ matrices A, B, suppose $B = PAQ$ for some $n \times n$ units P, Q, and let $U = \begin{bmatrix} A & \alpha \\ \beta & t \end{bmatrix}$ be an $(n+1) \times (n+1)$ completion of A. Consider the $(n+1) \times (n+1)$ (block-written) invertible matrices $\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$U' = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & \alpha \\ \beta & t \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} PAQ & P\alpha \\ \beta Q & t \end{bmatrix} = \begin{bmatrix} B & P\alpha \\ \beta Q & t \end{bmatrix}$$

is a completion for B.

(iii) Suppose $\gcd(A) \neq 1$ and so $\gcd(A) = d \notin U(R)$. Then $A = dA_1$ and so $B = dPA_1Q$, that is, $d \mid \gcd(B)$. Hence also $\gcd(B) \neq 1$.

(iv) Straightforward, by determinant expansion. □

By $\mathrm{diag}(a_1, ..., a_n)$ we denote a *diagonal* $n \times n$ matrix with the (diagonal) entries $a_1, ..., a_n$. Denote by π the product of all the diagonal entries and by $\alpha_i = \dfrac{\pi}{a_i}$, $1 \leq i \leq n$, the products of $n-1$ of these.

We can now prove the following

Theorem 11. *A diagonal $n \times n$ matrix $\mathrm{diag}(a_1, ..., a_n)$ over a Bézout ring R is completable iff all the products α_i, $1 \leq i \leq n$, are (collectively) coprime.*

Proof. If $\gcd(\alpha_1, \alpha_2, ..., \alpha_n) = 1$, a completion for $\mathrm{diag}(a_1, ...a_n)$ is of form $U = \begin{bmatrix} a_1 & 0 & \cdots & 0 & c_1 \\ 0 & a_2 & \cdots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n & c_n \\ b_1 & b_2 & \cdots & b_n & 0 \end{bmatrix}$. Expanding $\det(U)$ along the last row gives

$$(-1)^n \det(U) = b_1c_1\alpha_1 - b_2c_2\alpha_2 + ... + (-1)^n b_nc_n\alpha_n.$$

Since the α_i's are (collectively) coprime there exist elements β_i, $1 \leq i \leq n$ such that $\sum_{i=1}^{n} \beta_i\alpha_i = 1$. It suffices to choose randomly b_i, c_i such that $\beta_i = (-1)^{i+1}b_ic_i$.

Conversely, if $\det(U) = 1$ then the $n-1$ products α_i, $1 \le i \le n$, are (collectively) coprime. □

Corollary 12. *Let R be a (Henriksen) elementary divisor ring. An $n \times n$ matrix A over R has an invertible $(n+1) \times (n+1)$ completion iff in a diagonal reduction of A, all the products of $n-1$ diagonal entries are (collectively) coprime.*

Examples. 1) For $n = 2$, if $gr + hs = 1$ (i.e. r and s are coprime) then $\begin{bmatrix} r & 0 & -1 \\ 0 & s & -1 \\ h & g & 0 \end{bmatrix}$ and $\begin{bmatrix} r & s & 0 \\ 0 & 0 & 1 \\ h & -g & * \end{bmatrix}$ are completions over any ring.

2) It is well-known that every nontrivial 2×2 idempotent matrix over any Bézout domain is similar to E_{11}. Hence, every nontrivial 2×2 idempotent matrix over any Bézout domain is completable, since E_{11} is obviously completable.

In detail, if $E = \begin{bmatrix} x & y \\ z & 1-x \end{bmatrix}$ is a nontrivial idempotent, the similarity $PE = E_{11}P$ is given by $P = \begin{bmatrix} d & y' \\ -z' & x' \end{bmatrix}$ where $d = \gcd(x, z)$, $x = dx'$, $z = dz'$and $y = x'y'$ (because $yz' = x'(1-x)$ and $\gcd(x', z') = 1$ imply $x' \mid y$). Therefore, for a completion of E we start with a completion of E_{11} (say) $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ and compute $U' = \begin{bmatrix} P^{-1} & 0 \\ 0 & 1 \end{bmatrix} U \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}$ for the P above. The completion is $U' = \begin{bmatrix} x & y & -y' \\ z & 1-x & d \\ z' & -x' & 0 \end{bmatrix} = \begin{bmatrix} dx' & x'y' & -y' \\ dz' & y'z' & d \\ z' & -x' & 0 \end{bmatrix}$ since $\det(U') = (dx' + y'z')^2 = 1^2 = 1$.

3) Since over any integral domain, 2×2 nilpotent matrices are of form $\begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ with $x^2 + yz = 0$, the only matrix completions are $\begin{bmatrix} x & \pm 1 & 0 \\ \mp x^2 & -x & \mp 1 \\ -1 & 0 & 0 \end{bmatrix}$ and transposes.

Remark. A product of two completable matrices may not be completable: just take $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$.

Question. Can we prove the same completion theorems over a Hermite, or over a Bézout ring ?

5. Unit stable range one for a 3 × 3 nilpotent matrix

In this section, for any elementary divisor ring R, we show that *the* 3×3 *nilpotent matrix* $T =: \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ *has unit stable range one in* $\mathbb{M}_3(R)$.

First notice that T is equivalent to $2E =: 2(E_{11} + E_{22}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so it suffices to check the unit sr1 condition for $2E$.

Next, notice that for unit sr1 elements, the sr1 condition may be simplified as follows: for every x there is a unit y such that $(y+x)a-1 \in U(R)$.

Also recall that *unit sr1 is invariant to equivalences* and *any multiple of E_{ij} has unit sr1 in any* $\mathbb{M}_n(R)$ (e.g. reconsider the proof from [2]).

Hence $2E_{11}, 2E_{22}$ have unit sr1 in $\mathbb{M}_n(\mathbb{Z})$. However $2I_3$ has **not** (even) sr1 in $\mathbb{M}_3(\mathbb{Z})$.

Actually we can prove more

Proposition 13. *An integral scalar matrix $A = nI_3$ has sr1 iff $n \in \{-1, 0, 1\}$.*

Proof. Using equivalences, it is easy to see that $\text{diag}(r,s,t)$ has sr1 iff $\text{diag}(t,s,r)$ has sr1, and, $\text{diag}(r,s,t)$ has sr1 iff $\text{diag}(r,-s,t)$ has sr1. Therefore, when dealing with integral diagonal matrices $\text{diag}(n,m,l)$, with respect to sr1, we can suppose $0 \leq n, m, l$.

Suppose $1 \leq n$. For every multiple of I_3, we have to indicate an X for which no Y exists such that $A + Y(XA - I_3)$ has ± 1 determinant.

Since for $n = 1$, I_3 is a unit, we take $A = nI_3$ for $n \geq 2$ and consider $X = -(n^2+1)I_3$. Then $Y(XA - I_3) = -(1+n+n^3)Y$ and we can compute the determinant in the factor ring $\mathbb{Z}/(1+n+n^3)\mathbb{Z}$. The characterization becomes n^3 congruent to ± 1 mod $(1+n+n^3)$, which is impossible since $n \geq 2$. Hence multiples nI_3 with $n \geq 2$ have not sr1. □

We are now ready to prove the main result of this section

Proposition 14. *Let R be an elementary divisor ring. The matrix $A = 2E$ has unit sr1 in $\mathbb{M}_3(R)$.*

Proof. As mentioned above, for every 3×3 matrix X we will find a 3×3 matrix Y such that if $S = X + Y$, we have $\det(SA - I_3) \in \{\pm 1\}$.

Since $SA - I_3 = \begin{bmatrix} 2s_{11}-1 & 2s_{12} & 0 \\ 2s_{21} & 2s_{22}-1 & 0 \\ 2s_{31} & 2s_{32} & -1 \end{bmatrix}$ we get

$$-\det(SA - I_3) = \det \begin{bmatrix} 2s_{11}-1 & 2s_{12} \\ 2s_{21} & 2s_{22}-1 \end{bmatrix} = 4(s_{11}s_{22} - s_{12}s_{21}) - 2(s_{11}+s_{22}) + 1.$$

Hence, for every 2×2 matrix X we will find a 2×2 matrix Y (not necessarily invertible, but with coprime entries) such that $4\det(Y+X) - 2\text{Tr}(Y+X) + 1 \in \{\pm 1\}$.

This suffices because, if with coprime entries, Y can be completed to an invertible 3×3 matrix, using the result in the previous section.

For any $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ we choose $Y = \begin{bmatrix} -x_{11} & 1 \\ -x_{21} & -x_{22} \end{bmatrix}$, which clearly has coprime entries. Then $\det(X+Y) = 0 = \text{Tr}(X+Y)$ and so $4\det(X+Y) - 2\text{Tr}(X+Y) + 1 = 1$. □

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