SIMILARITY FOR ZERO-SQUARE MATRICES

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ABSTRACT. We show that an $n \times n$ zero-square matrix over a commutative unital ring R is similar to a multiple of E_{1n} if R is a Bézout domain and $n = 2$, 3, but there are zero-square matrices which are not similar to any multiple of E_{1n} whenever $n \geq 4$, over any commutative unital ring. As a consequence, for $n = 2$, 3 such matrices have stable range one.

1. Introduction

An integral domain is a *GCD* domain if every pair a, b of nonzero elements has a greatest common divisor, denoted by $\gcd(a, b)$ and a *Bézout* domain if $\gcd(a, b)$ is a linear combination of a and b. GCD domains include unique factorization domains, Bézout domains and valuation domains. If $\gcd(a, b) = 1$ we say that a and b are *coprime*.

It is not hard to prove that *every zero-square* 2×2 *matrix over a Bézout domain* R *is similar to* rE_{12}*, for some* $r \in R$ (see Section 3).

The aim of this paper is to extend the above result for zero-square 3×3 matrices over Bézout domains and to show that the property cannot be extended for $n \times n$ zero-square matrices if $n \geq 4$. That is, we prove the following

Theorem. *Let* R *be a Bézout domain. Every zero-square matrix of* $\mathbb{M}_3(R)$ *is similar to* rE_{13} *for some* $r \in R$.

Theorem. *Over any commutative ring and for every* $n \geq 4$*, there are zero-square* $n \times n$ *matrices which are not similar to multiples of* E_{1n}.

In our extension we have to solve a special type of completion problem: two unimodular 3-rows are given with some additional properties and we are searching for a completion to an invertible 3×3 matrix.

In Section 2, general results on zero-square $n \times n$ matrices are proved together with second theorem above.

For the sake of completeness, Section 3 covers the zero-square 2×2 case. In Section 4 we prove the first theorem above, that is, we settle the 3×3 zero-square case. Since multiples of E_{ij} are known to have stable range one, a consequence of our results is that zero-square 2×2 and 3×3 matrices over any Bézout domain (in particular over the integers) have stable range one.

E_{ij} denotes the $n \times n$ matrix with all entries zero excepting the (i, j) entry which is 1. By 0_n we denote the zero $n \times n$ matrix. For a square matrix A over a *commutative* ring R, the determinant and trace of A are denoted by $\det(A)$ and $\mathrm{Tr}(A)$, respectively. For a matrix A, $\gcd(A)$ denotes the greatest common divisor of all the entries of A. For a unital ring R, $U(R)$ denotes the set of all the units of R.

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2. Zero-square $n \times n$ matrices

In order to describe the zero-square $n \times n$ matrices over commutative (unital) rings or over integral domains, denote by T_{ab}^{cd} the 2×2 minor on the rows a and b and on the columns c and d.

Proposition 2.1. *Let $T = [t_{ij}]_{1 \leq i,j \leq n}$ be an $n \times n$ matrix over a commutative ring R and let $t_{ij}^{(2)}$ be the entries of T^2. Then*

$$\begin{array}{ccc} t_{ij}^{(2)} & = & t_{ij}\mathrm{Tr}(T) + \sum_{k \in \{1,...,n\}-\{i,j\}} T_{ik}^{kj} & i \neq j \\ t_{ii}^{(2)} & = & t_{ii}\mathrm{Tr}(T) + T_{i1}^{1i} + ... + T_{i,i-1}^{i-1,i} + T_{i,i+1}^{i+1,i} + ... + T_{in}^{ni} & i = j \end{array}.$$

Proof. Simple computation of $\mathrm{row}_i(T) \cdot \mathrm{col}_j(T)$ for $i \neq j$ or $i = j$. □

First recall that the *rank* of a (not necessarily square) matrix A (denoted $\mathrm{rk}(A)$) can be defined *over any commutative ring* R, using the annihilators of the ideals $I_t(A)$ generated by the $t \times t$ minors of A (see e. g. [1]). In particular, $\mathrm{rk}(A) = 1$ if all 2×2 minors are zero and and these two condition are equivalent over integral domains. Then it can be shown that *equivalent matrices* (so, in particular, similar matrices) *have the same rank* (see [1], **4.11**).

Therefore

Corollary 2.2. *Let T be an $n \times n$ matrix over any commutative ring. If all 2×2 minors of T are zero and $\mathrm{Tr}(T) = 0$ then $T^2 = 0_n$.*

Remark. Over any integral domain a (well-known) converse also holds: *If $T^2 = 0_n$ then* $\det(T) = \mathrm{Tr}(T) = 0$.

Over any integral domain, in order to have a characterization of form

$$T^2 = 0_n \text{ if and only if } \mathrm{rk}(T) = 1 \text{ and } \mathrm{Tr}(T) = 0,$$

the only remaining implication is that, $T^2 = 0_n$ and $\mathrm{Tr}(T) = 0$ imply $\mathrm{rk}(T) = 1$ (i.e. all 2×2 minors of T vanish).

In what follows we show that this implication holds over a commutative ring for $n = 3$ if 2 is not a zero divisor, but fails for any $n \geq 4$.

Theorem 2.3. *Let R be a commutative unital ring such that 2 is not a zero divisor and let $T \in \mathbb{M}_3(R)$ with $\mathrm{Tr}(T) = 0$. Then $T^2 = 0_3$ if and only if all 2×2 minors of T equal zero.*

Proof. To avoid too many indexes and emphasize the diagonal elements (i.e. the zero trace) we write $T = \begin{bmatrix} x & a & c \\ b & y & e \\ d & f & -x-y \end{bmatrix}$.

If $\mathrm{Tr}(T) = 0$, the condition $T^2 = 0_3$ is *equivalent* to the following nine LHS equalities

$$\begin{array}{cc}
x^2 + ab + cd = 0 \quad (1) & \\
a(x+y) + cf = 0 \quad (2) & T_{13}^{23} = 0 \\
ae = cy \quad (3) & T_{12}^{23} = 0 \\
b(x+y) + de = 0 \quad (4) & T_{23}^{13} = 0 \\
y^2 + ab + ef = 0 \quad (5) & \\
bc = ex \quad (6) & T_{12}^{13} = 0 \\
bf = dy \quad (7) & T_{23}^{12} = 0 \\
ad = fx \quad (8) & T_{13}^{12} = 0 \\
(x+y)^2 + cd + ef = 0 \quad (9) &
\end{array}.$$

The two terms equalities (i.e., (3), (6), (7), (8)) are *equivalent* to the vanishing of four 2×2 minors. Just look at the RHS column of vanishing minors. Further, two other equalities, namely, (2) and (4), are *equivalent* to the vanishing of another two minors.

Thus, this equivalently covers the six *off diagonal* 2×2 minors. What remains are the vanishing of the three 2×2 *diagonal* minors.

From $x^2 + ab + cd = 0$, $y^2 + ab + ef = 0$ and $(x + y)^2 + cd + ef = 0$ we get (since 2 is not a zero divisor) $xy = ab$, and so another zero 2×2 minor. Finally using $x^2 + ab + cd = 0$, $y^2 + ab + ef = 0$ and $xy = ab$, we get the last two zero 2×2 diagonal minors: $x(x + y) + cd = 0$ and $y(x + y) + ef = 0$.

The converse was settled in the general $n \times n$ case in Corollary 2.2. □

Remark. The hypothesis "2 *is not a zero divisor*" is essential for the vanishing of the three diagonal 2×2 minors (over any commutative ring). In searching for an example (see example 5 below), the following observations gathered in the next lemma helped. We skip the easy proof of (i) and (ii).

Lemma 2.4. *Suppose* $T^2 = 0_3$ *and* $\mathrm{Tr}(T) = 0$.

(i) If any diagonal 2×2 *minor is zero, so are the other two diagonal* 2×2 *minors.*

(ii) If any entry of T *is not a zero divisor, then all* 2×2 *minors are zero.*

(iii) Then $\det(T) = 0$ *and for the diagonal* 2×2 *minors we have* $T_{12}^{12} t_{33} = 0$, $T_{13}^{13} t_{22} = 0$ *and* $T_{23}^{23} t_{11} = 0$, *that is, with the notations in the previous proof,* $(xy - ab)(x + y) = 0$, $(x(x + y) + cd)y = 0$ *and* $(y(x + y) + ef)x = 0$.

Proof. (iii) If $T^2 = 0_3$, clearly $T^3 = 0_3$ and $\mathrm{Tr}(T^2) = 0$. Replacing in Cayley-Hamilton's theorem, i.e.,

$$T^3 - \mathrm{Tr}(T)T^2 + \frac{1}{2}[\mathrm{Tr}^2(T) - \mathrm{Tr}(T^2)]T - \det(T)I_3 = 0_3,$$

gives now $\det(T)I_3 = 0$ and so $\det(T) = 0$. As for the diagonal minors, first recall (proof of Theorem 2.3) that in the given hypotheses, the off diagonal minors, vanish. Expanding the determinant along the third row gives $T_{12}^{12} t_{33} = 0$, i.e., $(xy - ab)(x + y) = 0$. The other two relations are obtained similarly by expanding the determinant along the second row and along the first row, respectively. □

Actually, for a 3×3 matrix over a commutative ring we can consider the following four conditions

(A) $T^2 = 0_3$, and

(B) all 2×2 minors of T equal zero,

(C) $\det(T) = 0$,

(D) $\mathrm{Tr}(T) = 0$.

The examples below show that excepting **(B)** $\Longrightarrow$ **(C)**, without other hypothesis, all the above listed conditions are (individually) logically independent. Among the necessary 12 examples we select some nontrivial ones.

Examples. 1) Over $\mathbb{Z}_4$ take $T = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Then $T^2 = \begin{bmatrix} 8 & 8 & 8 \\ 16 & 16 & 16 \\ 12 & 12 & 12 \end{bmatrix} = 0_3$ but $\mathrm{Tr}(T) = 2 \neq 0$ and the diagonal minors $T_{13}^{13} = T_{23}^{23} = 2 \neq 0$. Moreover, $T_{23}^{13} = 2 \neq 0$, a not diagonal minor. However $\det(T) = 0$. Hence **(A)** does not imply **(B)** nor **(D)**.

2) Consider $R = \mathbb{Z}_2[X, Y, Z]/I$ for $I := (X^2, Y^2, Z^2)$ and the diagonal matrix over R, $T = \begin{bmatrix} X+I & 0 & 0 \\ 0 & Y+I & 0 \\ 0 & 0 & Z+I \end{bmatrix}$. Then $T^2 = 0_3$ but the trace, all the diagonal 2×2 minors and the determinant are not zero. So **(A)** does not imply any of **(B)**, **(C)** or **(D)**.

3) Denote by E the matrix with all entries $= 1$. Then E satisfies **(B)** (and so **(C)**), but since $E^2 = 3E$, over $\mathbb{Z}_4$ (or $\mathbb{Z}_2$), $E^2 \neq 0_3$ and $\mathrm{Tr}(E) \neq 0$ that is, E does not satisfy **(A)** nor **(D)**.

4) A matrix can satisfy **(A)**, **(B)** and **(C)** without having zero trace (i.e., not **(D)**). Over $\mathbb{Z}_4$ the matrix $T_1 = 2I_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ has only zero 2×2 minors (and so zero determinant), satisfies $T^2 = 0$ but has $\mathrm{Tr}(T) = 2 \neq 0$.

5) Consider $R = \mathbb{Z}_2[X, Y]/I$ for $I := (X^2, Y^2)$ and the diagonal matrix over R, $T = \begin{bmatrix} X+I & 0 & 0 \\ 0 & Y+I & 0 \\ 0 & 0 & X+Y+I \end{bmatrix}$. Then $T^2 = 0_3$, $\mathrm{Tr}(T) = 0$, but the diagonal minors are not zero. Hence **(A)** and **(D)** do not imply **(B)**. Clearly, 2 is a zero divisor in R, and the example shows that the hypothesis added in Theorem 2.3 is *not* superfluous.

As noticed before, over an integral domain, for any $n \times n$ matrix, $T^2 = 0_n$ implies $\det(T) = 0$ and so $\mathrm{rk}(T) < n$. Therefore, for $n = 2$ and $T \neq 0_2$ clearly $\mathrm{rk}(T) = 1$, but for $n = 3$ this must be proved (as this was done in the previous theorem).

In closing this section, an example of 4×4 zero-square matrix (over any commutative unital ring) with zero trace and rank 2 is given below.

Example. $C_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}^2 = 0_4$, has zero trace but many not zero 2×2 minors (e.g. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, in the center).

Hence $T^2 = 0_4$ does not generally imply $\mathrm{rk}(T) = 1$. Adding to this example as many zero rows and columns as necessary, $T^2 = 0_n$ does not generally imply $\mathrm{rk}(T) = 1$, for any $n \geq 5$.

Since nonzero multiples of E_{1n} have rank 1, and similar matrices have the same rank, we obtain

Theorem 2.5. *Over any commutative unital ring and for every $n \geq 4$, there are $n \times n$ zero-square matrices which are not similar to any multiple of E_{1n}.*

3. The zero-square 2×2 case

To simplify the writing, some equalities below are used modulo *association* (in divisibility). For example, $a = b$ means $b = au$ for a unit u. Equivalently, $a \mid b$ and $b \mid a$.

The following lemmas list some well-known properties of a GCD domain.

Lemma 3.1. *Let R be a GCD domain with $a, b, c \in R$.*
1. $\gcd(ab, ac) = a \gcd(b, c)$.
2. *If* $\gcd(a, b) = 1$ *and* $\gcd(a, c) = 1$, *then* $\gcd(a, bc) = 1$.
3. *If* $\gcd(a, b) = 1$ *and* $a \mid bc$, *then* $a \mid c$.

Lemma 3.2. *Let R be a GCD domain and $b, c \in R$.*
1. $\gcd(b, c) = 1$ *implies* $\gcd(b^n, c) = 1$ *for any* $n \geq 1$.
2. *Let* $\gcd(b, c) = 1$. *If bc is a square, so are both b and c.*
3. $\gcd(a, b) = 1$ *and* $a \mid c$, $b \mid c$ *implies* $ab \mid c$.

Proof. 1. This follows from (2), the previous lemma.

2. Let $a^2 = bc$. Denote $b_1 = \gcd(b, a)$ and $c_1 = \gcd(c, a)$. Then $b = b_1 b_2$, $c = c_1 c_2$ and $a = b_1 x = c_1 y$ for some $b_2, c_2, x, y \in R$ with $\gcd(b_2, x) = 1 = \gcd(c_2, y)$. Since $\gcd(b, c) = 1$, it follows that $\gcd(b_i, c_i) = 1$, $i \in \{1, 2\}$.

From $a^2 = bc$ we get $b_1 c_1 xy = b_1 b_2 c_1 c_2$, whence $xy = b_2 c_2$. Using this as $x \mid b_2 c_2$ together with $\gcd(b_2, x) = 1$, we obtain $x \mid c_2$. Analogously we derive $y \mid b_2$ and conversely $b_2 \mid y$ and $c_2 \mid x$. Hence $x = c_2$, $y = b_2$.

Finally $b_1 c_2 = a = b_2 c_1$ used as in the previous two lines gives (together with $\gcd(b_i, c_i) = 1$, $i \in \{1, 2\}$) $b_1 = b_2$ and $c_1 = c_2$, as desired.

3. Write $c = aa' = bb'$. Since a, b are coprime, $a \mid b'$, i.e., $b' = ad$. Hence $c = bb' = abd$ and so $ab \mid c$. □

Notice that a zero-square 2×2 matrix over an *integral domain* R is of form $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ with $\alpha^2 + \beta\gamma = 0$. Indeed, let Q be the field of fractions of R. Then in $\mathbb{M}_2(Q)$, B is similar to qE_{12} for some $q \in Q$. So $\mathrm{Tr}(B) = 0$ and $\det(B) = 0$.

Proposition 3.3. *Every zero-square 2×2 matrix over a Bézout domain R is similar to rE_{12}, for some $r \in R$.*

Proof. The result is trivial for the zero matrix so we assume the matrix is not zero.

Take $T = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ and $x^2 + yz = 0$. We construct an invertible matrix $U = [u_{ij}]$ such that $TU = U(rE_{12})$ with a suitable $r \in R$.

Let $d = \gcd(x, y)$ and denote $x = dx_1$, $y = dy_1$ with $\gcd(x_1, y_1) = 1$. Then $d^2 x_1^2 = -dy_1 z$ and since $\gcd(x_1, y_1) = 1$ implies $\gcd(x_1^2, y_1) = 1$, it follows y_1 divides d. Set $d = y_1 y_2$ and so $T = \begin{bmatrix} x_1 y_1 y_2 & y_1^2 y_2 \\ -x_1^2 y_2 & -x_1 y_1 y_2 \end{bmatrix} = y_2 \begin{bmatrix} x_1 y_1 & y_1^2 \\ -x_1^2 & -x_1 y_1 \end{bmatrix} = y_2 T'$.

Since R is Bézout and $\gcd(x_1, y_1) = 1$, there exist $s, t \in R$ such that $sx_1 + ty_1 = 1$. Take $U = \begin{bmatrix} y_1 & s \\ -x_1 & t \end{bmatrix}$ which is invertible (indeed, $U^{-1} = \begin{bmatrix} t & -s \\ x_1 & y_1 \end{bmatrix}$). One can check $T'U = \begin{bmatrix} 0 & y_1 \\ 0 & -x_1 \end{bmatrix} = UE_{12}$, so $r = y_2$. □

Remark. We can write $T = r(UE_{12}U^{-1})$, so r is a common divisor of all the entries of T.

4. The zero-square 3×3 case

The following lemma and proposition will be useful for the extension of Proposition 3.3 to zero-square 3×3 matrices.

Lemma 4.1. *Let $a, b, c, a', b', c' \in R$, a GCD domain. If $ab' = a'b$, $ac' = a'c$, $bc' = b'c$ and the rows $\begin{bmatrix} a & b & c \end{bmatrix}$ and $\begin{bmatrix} a' & b' & c' \end{bmatrix}$ are unimodular then the pairs a, a', b, b' and c, c' are associated. Moreover, there exists a unit $u \in U(R)$ such that $\begin{bmatrix} a' & b' & c' \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} u$.*

Proof. Denote $\delta = \gcd(a, b)$ with $a = \delta a_1$, $b = \delta b_1$ and $\delta' = \gcd(a', b')$ and $a' = \delta' a'_1$, $b' = \delta' b'_1$. From $ab' = a'b$ cancelling $\delta\delta'$ we obtain $a_1 b'_1 = a'_1 b_1$. Since a_1, b_1 are coprime, it follows $a_1 \mid a'_1$. Symmetrically, since a'_1, b'_1 are coprime, it follows $a'_1 \mid a_1$, so that a_1, a'_1 are associates. Hence there is a unit $u \in U(R)$ such that $a_1 = a'_1 u$.

Further, notice that $\gcd(\delta, c) = \gcd(\gcd(a, b), c) = 1$ and so δ, c are coprime. Now we use $ac' = a'c$, that is, $\delta(a'_1 u)c' = \delta a_1 c' = \delta' a'_1 c$. Cancelling a'_1 we get $\delta u c' = \delta' c$ and since δ, c are coprime, $\delta \mid \delta'$. Symmetrically, $\delta' \mid \delta$ and so δ, δ' are also associates. Therefore $a = \delta a_1$ and $a' = \delta' a'_1$ are associates.

In a similar way, it follows that b, b' and c, c' are associates, respectively.

Finally, suppose $a' = au$, $b' = bv$ and $c' = cw$ for some $u, v, w \in U(R)$. From $ab' = a'b$ we get $abv = aub$, so $v = u$. Analogously, $w = v$ and so $w = v = u$, as claimed. □

Remark. The second hypothesis of the lemma can be stated as a matrix rank: $\text{rk} \begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix} = 1$.

Proposition 4.2. *Let R be a GCD domain. If $\text{rk} \begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix} = 1$ (i.e., $ab' = a'b$, $ac' = a'c$, $bc' = b'c$), $\delta = \gcd(a, b, c)$, $\lambda = \gcd(a', b', c')$ and $a = \delta a_1$, $b = \delta b_1$, $c = \delta c_1$, $a' = \lambda a'_1$, $b' = \lambda b'_1$ and $c' = \lambda c'_1$, then a_1, b_1, c_1 and a'_1, b'_1, c'_1 are respectively associated (in divisibility). Moreover, $\begin{bmatrix} a'_1 & b'_1 & c'_1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} u$ for some $u \in U(R)$.*

Proof. We just use the previous lemma. □

Next we need

Lemma 4.3. *Let R be a commutative unital ring and $\mathbf{a} = (a_1, a_2, a_3) \in R^3$. Then $\mathbf{a}$ is a unimodular row (in R) if and only if $\mathbf{a} + M \neq M$ for every maximal ideal M of R^3 (i.e., $\mathbf{a} + M \neq 0$ in R^3/M).*

Proof. Indeed, $\mathbf{a} \in R^3 - M$ is equivalent to at least one of $a_i \notin M$, $i \in \{1, 2, 3\}$. Equivalently, the system $\{a_1, a_2, a_3\} \nsubseteq M$. It suffices to notice that every proper ideal is (by Zorn's Lemma) included in a maximal ideal and, that an 3-vector is **not** a unimodular row (i.e., not an ideal generating system) if and only if the elements are included in a maximal ideal. □

Proposition 4.4. *Suppose $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$ are unimodular rows such that $\mathbf{a} \cdot \mathbf{b} = 1$, $\mathbf{a} \cdot \mathbf{c} = 0$. Then the cross product $\mathbf{b} \times \mathbf{c}$ is also a unimodular row.*

Proof. According to the previous lemma, it suffices to show that $\mathbf{b} \times \mathbf{c}$ is nonzero modulo any maximal ideal. Since R^3 is also a unital commutative ring, modulo any maximal ideal M, R^3/M is a field and $\mathbf{b} + M$, $\mathbf{c} + M$ are nonzero (otherwise these are not unimodular rows, i.e. ideal generating systems). It is easy to see that these two vectors are linearly independent (indeed, if $(\mathbf{a} + M)\cdot(\mathbf{b} + M) = 1 + M$, $(\mathbf{a}+M)\cdot(\mathbf{c}+M) = M$ and $\mathbf{b}+M = k(\mathbf{c}+M)$ then $1+M = M$, impossible). Hence their cross product is (well-known to be) nonzero and the proof is complete. □

Proposition 4.5. *Let R be a commutative unital ring and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be unimodular 3-vectors such that $\mathbf{a} \cdot \mathbf{b} = 1$ and $\mathbf{a} \cdot \mathbf{c} = 0$. There exists a unimodular 3-vector $\mathbf{x}$, also orthogonal on $\mathbf{a}$, such that $\mathbf{c} \cdot (\mathbf{x} \times \mathbf{b}) = 1$.*

Proof. By the above lemma, since $\mathbf{b}\times\mathbf{c}$ is also unimodular, there exists a unimodular n-vector $\mathbf{x}$ such that $\mathbf{x} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{x} \times \mathbf{b}) = 1$. If $\mathbf{a}\cdot\mathbf{x} = s$, replace $\mathbf{x}$ by $\mathbf{x} - s\mathbf{b}$ and then this vector is also orthogonal on $\mathbf{a}$ (indeed, $\mathbf{a} \cdot (\mathbf{x} - s\mathbf{b}) = \mathbf{a}\cdot\mathbf{x} - s(\mathbf{a}\cdot\mathbf{b}) = s - s = 0$). □

We are now ready to prove our main result

Theorem 4.6. *Zero-square 3×3 matrices over a Bézout domain, are similar to multiples of E_{13}.*

Proof. Again consider $T = \begin{bmatrix} x & a & c \\ b & y & e \\ d & f & -x-y \end{bmatrix}$ with $T^2 = 0_3$ (by Theorem 2.3, $\mathrm{rank}(T) = 1$, that is, all 2×2 minors are zero).

Denote $\delta = \gcd(x, a, c)$, $\lambda = \gcd(b, y, e)$ and $\gamma = \gcd(d, f, x+y)$ so that $x = \delta x_1$, $a = \delta a_1$, $c = \delta c_1$, $b = \lambda b_1$, $y = \lambda y_1$, $e = \lambda e_1$, $d = \gamma d_1$, $f = \gamma f_1$ and $x+y = \gamma(x_2+y_2)$.

According to Proposition 4.2, there are units u, v such that $\begin{bmatrix} b_1 & y_1 & e_1 \end{bmatrix} = \begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} u$ and $\begin{bmatrix} d_1 & f_1 & -x_2 - y_2 \end{bmatrix} = \begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} v$.

Hence $T = \begin{bmatrix} \delta x_1 & \delta a_1 & \delta c_1 \\ \lambda u x_1 & \lambda u a_1 & \lambda u c_1 \\ \gamma v x_1 & \gamma v a_1 & \gamma v c_1 \end{bmatrix}$ and since $\begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix}$ is unimodular, there are $s, t, z \in R$ and $s x_1 + t a_1 + z c_1 = 1$.

Note that $\mathrm{Tr}(T) = \delta x_1 + \lambda u a_1 + \gamma v c_1 = 0$.

Denote $r = \gcd(\delta, \lambda, \gamma) = \gcd(T)$ and denote $\delta = r\delta_1$, $\lambda = r\lambda_1$, $\gamma = r\gamma_1$. We are looking for an invertible matrix U such that $TU = U(rE_{13}) = \begin{bmatrix} 0 & 0 & ru_{11} \\ 0 & 0 & ru_{21} \\ 0 & 0 & ru_{31} \end{bmatrix}$.

Our choice for r is necessary: indeed, writing $T = rUE_{13}U^{-1}$, shows that r divides all entries of T. Also note that, if $\det(U) = 1$, every row and every column of U is unimodular.

We choose $\mathrm{col}_3(U) = \begin{bmatrix} s \\ t \\ z \end{bmatrix}$. By computation

$ru_{11} = \mathrm{row}_1(T) \cdot \mathrm{col}_3(U) = \delta(x_1 u_{13} + a_1 u_{23} + c_1 u_{33}) = \delta$,
$ru_{21} = \mathrm{row}_2(T) \cdot \mathrm{col}_3(U) = \lambda u(x_1 u_{13} + a_1 u_{23} + c_1 u_{33}) = \lambda u$,
$ru_{31} = \mathrm{row}_3(T) \cdot \mathrm{col}_3(U) = \gamma v(x_1 u_{13} + a_1 u_{23} + c_1 u_{33}) = \gamma v$, and,

$\begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} = 0$ and so

$\begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} U = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

Hence, the first column of U must be $\mathrm{col}_1(U) = \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} \frac{\delta}{r} \\ \frac{\lambda}{r}u \\ \frac{\gamma}{r}v \end{bmatrix}$. These fractions exist since $r = \gcd(\delta, \lambda, \gamma)$.

We indeed have $\begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} \begin{bmatrix} \frac{\delta}{r} \\ \frac{\lambda}{r}u \\ \frac{\gamma}{r}v \end{bmatrix} = \frac{1}{r}(\delta x_1 + \lambda u a_1 + \gamma v c_1) = \frac{1}{r}(x + y - (x+y)) = 0$ (because $\mathrm{Tr}(T) = 0$).

We are searching for a suitable column $\mathrm{col}_2(U)$ such that $U = \begin{bmatrix} \delta_1 & u_{12} & s \\ \lambda_1 u & u_{22} & t \\ \gamma_1 v & u_{32} & z \end{bmatrix}$ is invertible and $\begin{bmatrix} x_1 & a_1 & c_1 \end{bmatrix} U = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

Taking $\mathbf{a} = [x_1\ a_1\ c_1]$, $\mathbf{b} = [s\ t\ z]$ and $\mathbf{c} = [\delta_1\ \lambda_1 u\ \gamma_1 v]$ the existence of $\mathbf{x} = [u_{12}\ u_{22}\ u_{32}]$ follows (by transpose) from the previous proposition. □

Remarks. 1) Expanding $\det(U) = 1$ along the second column, we obtain for the unknown $\mathrm{col}_2(U)$ the system

$$\begin{array}{rcl} -(\lambda_1 uz - \gamma_1 vt)u_{12} + (\delta_1 z - \gamma_1 vs)u_{22} - (\delta_1 t - \lambda_1 us)u_{32} & = & 1 \\ x_1 u_{12} + a_1 u_{22} + c_1 u_{32} & = & 0 \end{array}.$$

The first equation (also) implies the unimodularity of $\mathrm{col}_2(U)$.

2) So far, an example of nilpotent matrix which has not stable range one was not found. The similarities proved for zero-square 2×2 and 3×3 matrices over Bézout domains, show that for such matrices a possible example should have nilpotent index 3.

Example. Now $x_1 = 6 = 2 \cdot 3$, $a_1 = 10 = 2 \cdot 5$, $c_1 = 15 = 3 \cdot 5$, so that no two of these are coprime.

$T_6 = \begin{bmatrix} -180 & -300 & -450 \\ 90 & 150 & 225 \\ 12 & 20 & 30 \end{bmatrix}$, $\delta = -30$, $\lambda = 15$, $\gamma = 2$ and so $r = 1$.

The second equation is a linear Diophantine equation, $6s + 10t + 15z = 1$. We denote $w = 3s + 5t$ and solve $2w + 15z = 1$. It gives $w = -7 + 15n$, $z = 1 - 2n$.

We choose $w = -7$ (for $n = 0$) and solve $3s + 5t = -7$. This gives for instance $s = -14$, $t = 7$, so we choose also $z = 1$ and $U = \begin{bmatrix} -30 & u_{12} & -14 \\ 15 & u_{22} & 7 \\ 2 & u_{32} & 1 \end{bmatrix}$.

Now the first equation is

$$-u_{12} \begin{vmatrix} 15 & 7 \\ 2 & 1 \end{vmatrix} + u_{22} \begin{vmatrix} -30 & -14 \\ 2 & 1 \end{vmatrix} - u_{32} \begin{vmatrix} -30 & -14 \\ 15 & 7 \end{vmatrix} = 1,$$

that is, $-u_{12} - 2u_{22} = 1$.

Hence $2u_{22} = -1 - u_{12}$ and so $6u_{12} - 5 - 5u_{12} + 15u_{32} = 0$ or $u_{12} + 15u_{32} = 5$. We can choose $u_{12} = 5$, $u_{32} = 0$ and so $u_{22} = -3$.

Indeed $\begin{bmatrix} -180 & -300 & -450 \\ 90 & 150 & 225 \\ 12 & 20 & 30 \end{bmatrix} \begin{bmatrix} -30 & 5 & -14 \\ 15 & -3 & 7 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -30 \\ 0 & 0 & 15 \\ 0 & 0 & 2 \end{bmatrix}$, as desired.

In closing, we state an

Open question. Find *explicitly* the invertible matrix $U = \begin{bmatrix} \delta_1 & u_{12} & s \\ \lambda_1 u & u_{22} & t \\ \gamma_1 v & u_{32} & z \end{bmatrix}$ such that $TU = U(rE_{13})$ with $r = \gcd(T)$.

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References

[1] W. C. Brown *Matrices over commutative rings*. Marcel Dekker, Inc. New York, Basel, Hong Kong, 1993.