

Rings with very few nilpotents

Grigore Călugăreanu

Abstract We show that any indecomposable ring with only 3 nonzero nilpotent elements, pairwise not commuting, is isomorphic to the ring of 2×2 matrices over the 2-element field. Rings with only one or two nonzero nilpotent elements are also investigated.

Keywords indecomposable ring · nilpotent element · finite ring · nilpotent radical

Mathematics Subject Classification (2010) 16N40 · 16U99 · 16N60

1 Introduction

In the sequel, R denotes an associative ring with identity, $N(R)$ the set of nilpotent elements of R (hereafter abbreviated *nilpotents*), $Nil^*(R)$ the upper nil radical of R , i.e. the sum of all nil ideals of R , $J(R)$ the Jacobson radical of R , $S(R)$ the set of zero-square elements of R , \mathbf{F}_p the field with p elements and for any ring S , $\mathcal{M}_n(S)$ denotes the ring of all $n \times n$ matrices with entries in S .

For unexplained notions and Ring Theory results we refer to [5]. For instance, (10.16) refers to chapter 10 of this book, namely Proposition (10.16).

An important line of study for finite rings is to establish how many nonisomorphic rings with a given order do exist (see e.g. [8]).

Already tackled in the seminal treatise of László Fuchs (see 129 in [2]), another line of study is to determine rings with given group of units, more recently revisited for finite rings by D. Dolžan (see [1]).

In this short note, we open another line of study: find all the nonisomorphic rings (with identity) which have a given finite number of nilpotents. The results we prove are especially encouraging to do so.

Among the $2^4 = 16$ elements of the matrix ring $\mathcal{M}_2(\mathbf{F}_2)$, there are exactly four nilpotents, the zero matrix and another three. It is readily checked that pairwise, these nonzero nilpotents do not commute.

Our main result shows that a converse also holds, that is

Theorem 1.1 *Let R be a unital (associative) ring which has exactly 3 nonzero nilpotents, pairwise not commuting, and has no nontrivial central idempotents. Then $R \cong \mathcal{M}_2(\mathbf{F}_2)$.*

We explain at once why the second hypothesis is necessary. Observe that $N(R \times S) = N(R) \times N(S)$ and that $|N(R)| = 1$ iff R is reduced. Therefore if a ring R has the property above, so is $R \times S$, for any reduced ring S . Hence we are actually interested in indecomposable rings with this property. Just recall that a ring is indecomposable iff it has no nontrivial central idempotents.

The last section contains similar results for two or three nilpotents. Surprisingly, these cases turn out to be more difficult than the four nilpotents case.

2 Only 3 not commuting nilpotents

The following result proved by C. Lanski in his paper [6], which has a very similar title to ours ($S(R)$ denotes the set of zero-square elements), will be useful.

Theorem 2.1 *If $S(R)$ is finite then: (i) $N(R)$ is finite; (ii) $S(R)/Nil^*(R)$ is finite; (iii) $R/Nil^*(R) \cong A \oplus B$ with A and B ideals, A reduced and B finite. More, in this case all these statements are equivalent.*

Next we give the

Proof of the Theorem 1.1.

Denote by $N(R) = \{0, t, t', t''\}$, the four nilpotents.

Step 1. (i) *The nilpotents are zero-square.*

Just observe that t, t^2 are commuting nilpotents (different, because the only nilpotent idempotent in any ring is 0), so only $t^2 = 0$ is possible (similarly for t' and t'').

(ii) $2t = 2t' = 2t'' = 0$.

Indeed $t \neq t + t$ and $2t$ is a nilpotent which commutes with t . So $2t = 0$ (and similarly for t' and t'').

(iii) *Sums of different nonzero nilpotents are not nilpotents.*

Since $t + t' = 0$ is not possible (otherwise $t' = -t$ commutes with t), and obviously $t + t' \notin \{t, t'\}$, suppose $t'' = t + t'$ with $t^2 = t'^2 = t''^2 = 0$. Then $0 = t''^2 = tt' + t't$ and by right multiplication with t this gives $tt't = 0$, so tt' is also zero-square. Since by (ii), $2tt' = 0$, we obtain $tt' = t't$, a contradiction.

(iv) *R has no nonzero nil ideals.*

We check that no subset of $N(R)$ containing 0 is an ideal.

(a) $\{0, t\}$ is (by (i) and (ii)) a subring but not an ideal. Indeed, we show that the product $tt' \notin \{0, t\}$. If $tt' = 0$ then $(t + t')^2 = t't$ and so $(t + t')^4 =$

$t'(tt')t = 0$, which contradicts (iii). If $tt' = t$ we get $t(1 - t') = 0$ and $t = 0$ (because $1 - t'$ is a unit), a contradiction. Similarly for $\{0, t'\}$ or $\{0, t''\}$.

(b) That $\{0, t, t'\}$ is not closed under multiplication, follows from the argument in (a). Similarly for the other two combinations.

(c) $N(R)$ is not closed under multiplication: we just have to show that the remaining possibility, $tt' = t''$, leads to a contradiction.

Indeed, multiplying this equality by t to the left, gives $tt'' = 0$, which is dealt as in (a). Hence the subsets in (b) and (c) are not even subrings.

Step 2. According to (iv) above, $Nil^*(R) = (0)$.¹

From the above Lanski Theorem (iii), since R is supposed indecomposable and not reduced, it follows that R is finite (and so clearly left Artinian). According to (10.16), R is semiprime and by (10.24), it is semisimple. Using Wedderburn-Artin Theorem (3.5), again since R is indecomposable, it follows that R is simple (Artinian) and so isomorphic to some $\mathcal{M}_n(D)$ for a positive integer n and a division ring D .

Step 3. If $n \geq 3$ and $|D| \geq 2$, then $\mathcal{M}_n(D)$ has at least $2^3 = 8$ strictly upper triangular matrices which are known to be nilpotent.

Therefore only $|D| = 2$ and $n = 2$ suits our hypothesis, and the proof is complete.

Remarks. 1) Actually, since R is finite and indecomposable, the order $|R|$ of R must be a power of a prime. In our case this is 2^4 .

2) If R is a finite simple ring, then it is isomorphic to a matrix ring $\mathcal{M}_n(\mathbf{F}_p)$ over the field with p elements, for a suitable prime p .

3 Rings with two or three nilpotents

As explained in the Introduction, here too, we consider only indecomposable rings.

First suppose $N(R) = \{0, t\}$, that is, R has only one nonzero nilpotent. Arguing as in the previous section, $t^2 = 0 = 2t$, and so $N(R)$ is a 2-element subring of R . There are two possibilities: $N(R)$ is a (nil) ideal or not.

If it is not, $Nil^*(R) = (0)$ and using Step 2 from the previous Section, R is isomorphic to some $\mathcal{M}_n(D)$ for a positive integer n and a division ring D . Since the smallest such matrix ring has 3 nilpotents, this is not our case. Therefore $N(R) = Nil^*(R) = \{0, t\} \neq (0)$ must be a (nil) ideal of R . Rings with $N(R) = Nil^*(R)$ were called NI by Marks (see [7]). Note that R is NI iff $N(R)$ forms an ideal iff $R/Nil^*(R)$ is reduced. Notice that, in this case, the ring might not be finite.

Example. Let R be any ring and let M be any (R, R) -bimodule. Recall that the *trivial extension* of R by M is the ring $T(R, M)$ defined on $R \times M$ by the usual addition and the multiplication $(r, m) * (r', m') = (rr', rm' + r'm)$. The trivial extension $T(\mathbf{Z}, \mathbf{Z}_{p^k})$ with $k = 2$ and $p = 2$ or $p = 3$, are infinite indecomposable NI rings with exactly two and three nilpotents, respectively.

¹ Such rings were called *nil-semisimple* in [4].

If the ring is finite then it can be determined by the Proposition below.

Denote by $\mathcal{T}_{m+1}(\mathbf{F}_p) = \{(a_{ij}) \in \mathcal{M}_{m+1}(\mathbf{F}_p) \mid a_{21} = a_{31} = \dots = a_{m+1,1} = 0\}$, the matrix subring and recall the following result obtained by Hirano, Sumiyama in [3].

Theorem 3.1 *Let R be a directly indecomposable finite ring with $|J(R)| = p^m$, where p is a prime and m is a positive integer. Then $p^{m+1} \leq |R| \leq p^{m^2+m+1}$. The first equality holds iff $R/J(R) = \mathbf{F}_p$ and the second equality holds iff R is either isomorphic or antiisomorphic to the matrix subring $\mathcal{T}_{m+1}(\mathbf{F}_p)$.*

Notice that for $m = 1$ and $p = 2$, $\mathcal{T}_2(\mathbf{F}_2)$ is the (sub)ring of upper triangular matrices over \mathbf{F}_2 .

Proposition 3.2 *A finite indecomposable ring has only one nonzero nilpotent element iff it is isomorphic to any of the following rings: \mathbf{Z}_4 , $\mathbf{Z}_2[i]$, $\mathcal{T}_2(\mathbf{F}_2)$ or the dual of $\mathcal{T}_2(\mathbf{F}_2)$.*

Proof. Since $\widehat{2}$ is the only nonzero nilpotent of \mathbf{Z}_4 , $1+i$ is the only nonzero nilpotent of $\mathbf{Z}_2[i]$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is the only nonzero nilpotent of $\mathcal{T}_2(\mathbf{F}_2)$ and nilpotents are preserved by antiisomorphisms, we only have show the conditions are necessary. Since we supposed R to be indecomposable, as is well-known, $|R|$ is a power of a prime p . Further, since R is finite, $N(R) = Nil^*(R) = J(R) = \{0, t\}$ (see e.g. (10.27)), and so this prime is $p = 2$.

Finally using the Theorem above (for $m = 1$), it follows that $|R| = 4$, or $|R| = 8$. In the first case, among the four (nonisomorphic) unital rings, only the rings in the statement are known to have only one nonzero nilpotent, and in the second case, $\mathcal{T}_2(\mathbf{F}_2)$, the (sub)ring of upper triangular matrices, or its dual, have exactly one nonzero nilpotent. \square

Remark. The ring $\mathbf{Z}_2[i]$ may also be presented as a subring of $\mathcal{M}_2(\mathbf{F}_2)$, namely $\{0_2, I_2, U, I_2 + U\}$, with unit $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and zero-square $I_2 + U$.

Next suppose $N(R) = \{0, t, t'\}$, i.e. R has only two nonzero nilpotents. Then two cases would be possible: t, t' do not commute, or $tt' = t't$. In the next proof we show that only the second situation may occur.

Notice that the matrix subring in Hirano, Sumiyama Theorem above is now $\mathcal{T}_3(\mathbf{F}_3) = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \right\}$, and has at least $2^4 = 16$ nonzero nilpotents (the strictly triangular matrices).

The results are gathered in the following

Proposition 3.3 *If an indecomposable ring R has only two nonzero nilpotents, these nilpotents must commute and the ring is NI. If in addition, the ring is finite, then either $R \cong \mathbf{Z}_9$ or $R \cong \mathbf{Z}_3[i]$ (and conversely).*

Proof. If $tt' \neq t't$, as in Step 1 (iv) we show that R has no nonzero nil ideals (indeed $tt' = t$ implies $t = 0$; for $N(R)$, if $tt' = 0$ then $t't \neq 0$ so we reduce to the previous case), that is $\text{Nil}^*(R) = (0)$. Step 2 and Step 3 apply and lead to at least 3 nonzero (not commuting) nilpotents, which is not our case. Therefore, we can continue only with $tt' = t't$.

In this case, their sum and product are also nilpotents, and so $t + t' = 0 = tt'$. Thus $t' = -t$ and again $t^2 = 0 = t'^2$. As in the proof of the previous Proposition, $N(R)$ must be a (nil) ideal and so $N(R) = \text{Nil}^*(R) = \{0, t, t'\}$. Hence R is again NI.

If R is finite (indecomposable) we again apply Hirano, Sumiyama Theorem ($m = 1$) for the conclusion $|R| = 3^2$ (now $|R| = 3^3$ is not suitable, as observed in the paragraph before the Proposition), and so R is either isomorphic to \mathbf{Z}_9 or $\mathbf{Z}_3[i]$. \square

Acknowledgements Thanks are due to Greg Marks for a useful reminder.

References

1. DOLŽAN, D. – *Group of units in a finite ring*, J. Pure Appl. Algebra **170** (2002), no. 2-3, 175-183.
2. FUCHS, L. – *Infinite abelian groups*, Vol. II, Pure and Applied Mathematics. Vol. 36-II, Academic Press, New York-London, 1973, ix+363 pp.
3. HIRANO, Y.; SUMIYAMA, T. – *On orders of directly indecomposable finite rings*. Bull Austral. Math. Soc. **46** (1992), no. 3, 353-359.
4. KIM, N.K.; LEE, Y.; RYU, S.J. – *An ascending chain condition on Wedderburn radicals*, Comm. Algebra **34** (2006), no. 1, 37-50.
5. LAM, T.Y. – *A first course in noncommutative rings*, Second edition. Graduate Texts in Mathematics, 131, Springer-Verlag, New York, 2001, xx+385 pp.
6. LANSKI, C. – *Rings with few nilpotents*, Houston J. Math. **18** (1992), no. 4, 577-590.
7. MARKS, G. – *On 2-primal Ore extensions*, Comm. Algebra **29** (2001), no. 5, 2113-2123.
8. SLOANE, N.J.A. – *The on-line encyclopedia of integer sequences*, <https://oeis.org/A027623>.

Received: 19.II.2016 / Accepted: 10.I.2017

AUTHOR

GRIGORE CĂLUGĂREANU,
 Department of Mathematics,
 Babeş -Bolyai University,
 1 Kogălniceanu street,
 Cluj-Napoca, România,
E-mail: calu@math.ubbcluj.ro