An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.)
Manuscript No. (will be inserted by the editor)

Are nil-clean 3×3 integral matrices, exchange ?

Grigore Călugăreanu · Horia F. Pop

Abstract We expose multiple efforts made trying to give an answer to the question in the title. Some of these were possible only with computer aid. Summarizing, the authors belief is that the question has an affirmative answer.

Keywords exchange · nil-clean · clean · 3×3 integral matrix · similarity · diagonal reduction

Mathematics Subject Classification (2010) MSC 16U10 · MSC 16U60 · 11D04 · 11D09 · 11-04 · 15B36

1 Introduction

An element a in a unital ring R is called *nil-clean* ([5]) if $a = e + t$ with idempotent e and nilpotent t, and is called *clean* ([7]) if $a = e + u$ with idempotent e and unit u. Since nil-clean rings are clean ([5]) a natural question stated in [5] (2006) was whether there are nil-clean elements in a ring which are not clean. It took some time but in [1] (2014) a 2×2 integral matrix was given as an example of *nil-clean matrix which is not clean*.

Also in [7] (1977) *exchange* elements (called *suitable*) were defined in four equivalent ways. One one these is: an element a in a ring R is (left) *exchange* if there is an idempotent e such that $e - a \in R(a - a^2)$. It was proved in [8] that every left exchange element is also right exchange and conversely. Since clean rings are exchange ([7]), we can ask another natural question: *are there nil-clean elements which are not exchange* ?

Recall that a (nil-)clean element is called *trivial* if its decomposition uses a trivial idempotent (i.e. 0 or 1). Notice that the trivial nil-clean elements are the nilpotents or the unipotents ($1 + t$ with nilpotent t), which are clean and so exchange. Hence in searching for an example of nil-clean element which is not exchange we may discard the trivial nil-clean elements.

In the last 3-4 years the following result was circulating in the Ring Theory community: *Let R be any ring, $a \in R$, and suppose that $a = e + q$ where $e^2 = e$ and $q^2 = 0$. Then a is exchange in R.*

Therefore, *nil-clean 2×2 matrices over any commutative domain are exchange.*

Recently, the (above) condition $q^2 = 0$ was improved (see [3]): if $e^2 = e$, q is nilpotent and $eq^2 = qeq$ or $q^2e = qeq$ then $e + q$ is exchange. Such elements were called *medium nil-clean*.

By the above mentioned result, looking for an example of nil-clean integral matrix which is not exchange, $\mathcal{M}_2(\mathbb{Z})$ and subrings of this won't do. Before passing to $\mathcal{M}_3(\mathbb{Z})$, several other attempts can be made, into special subrings of $\mathcal{M}_3(\mathbb{Z})$. These were unsuccessful and are roughly discussed in section 2.

Hence it is reasonable to search such an example in $\mathcal{M}_3(\mathbb{Z})$.

There are two main difficulties in searching for a 3×3 integral nil-clean matrix which is not exchange.

If we start with some nil-clean matrix *it is hard to prove* it is not exchange and, if we start with some not exchange matrix *it is hard to show* it is nil-clean. Computer aid helps but cannot give a firm answer.

In this note we expose our attempts in trying to answer the question in the title. In doing so, we also obtain some partial results, results which, together with several other attempts made by computer, seem to support the affirmative answer to this question.

To simplify what follows we state the

Conjecture 1.1 Nil-clean 3×3 integral matrices are exchange.

In Section 3, using a completion theorem proved in [2], and similarity invariance, we describe the simplest procedure which constructs, up to similarity, all nil-clean matrices and verifies if these are exchange or not.

In Section 4, we state and prove some positive results regarding this conjecture.

In Section 5, a different attempt was made: starting with some (large classes of) not exchange 3×3 matrices A, we list all the 3×3 nilpotents T, and check whether at least one difference $A - T$ is idempotent. None of all these verifications gave an idempotent difference.

Finally, Section 6 describes several computer programs and their results, based on Sections 3, 4 and 5, which, to some extent, support the affirmative answer for our conjecture.

Summarizing, the authors strongly believe that the conjecture has an affirmative answer.

As stated above, an element a in a ring R is exchange if there exists an element $m \in R$ such that $a + m(a - a^2)$ is idempotent. To simplify the wording m with be called an *exchanger* for a. We mention that something analogous exists for exchange elements (see [6]): x is called a *suitabilizer* for a via the idempotent f if $xa - fx = 1$. However, since the set of suitabilizers and the set of exchangers may differ, for a given (exchange) element a, we will use the term exchanger in this paper.

All rings we consider are (associative and) unital.

2 Subrings of $\mathcal{M}_3(\mathbb{Z})$

Since the desired counterexamples cannot be found in $\mathcal{M}_2(\mathbb{Z})$ or subrings, that is, 4 (integer) variables are not sufficient, another attempt could be made with *5 variables*, namely with a subring of $\mathcal{M}_3(\mathbb{Z})$ (*cross* 3×3 *matrices*), $V = \left\{ \begin{bmatrix} a & 0 & b \\ 0 & e & 0 \\ c & 0 & d \end{bmatrix} | a, b, c, d, e \in \mathbb{Z} \right\}$, which is ring isomorphic to $\mathcal{M}_2(\mathbb{Z}) \times \mathbb{Z}$.

Since (up to isomorphism) this is a direct product, a pair $(r, s) \in R \times S$ is idempotent or nilpotent or nil-clean or exchange iff so are both components. Since exchange (or clean) integers are only $\{-1, 0, 1, 2\}$ and nil-clean are only $\{0, 1\}$, that is, nil-clean integers are (clean and) exchange, according to the result mentioned in the introduction, nil-clean matrices in V are exchange.

Next, the natural candidate with *6 variables* is the subring $\mathcal{T}$ of (upper) triangular matrices of $\mathcal{M}_3(\mathbb{Z})$. We can prove

Proposition 2.1 *Nil-clean matrices in $\mathcal{T}$ are exchange.*

Proof. It is easy to see that nontrivial (i.e. different from 0_3, I_3) idempotents in $\mathcal{T}$ are of form $E_1 = \begin{bmatrix} 0 & b & be \\ 0 & 1 & e \\ 0 & 0 & 0 \end{bmatrix}$ (if the trace is 1), or $E_2 = \begin{bmatrix} 1 & b & -be \\ 0 & 0 & e \\ 0 & 0 & 1 \end{bmatrix}$ (if the trace is 2), and, nilpotents are precisely the strictly (0 on the diagonal) upper triangular matrices (the Jacobson radical of $\mathcal{T}$). Accordingly, *nontrivial nil-clean matrices in $\mathcal{T}$* have trace $= 1$ or 2, determinant $= 0$ and have the form $A_1 = \begin{bmatrix} 0 & b+\beta & be+\gamma \\ 0 & 1 & e+\varepsilon \\ 0 & 0 & 0 \end{bmatrix}$ or, $A_2 = \begin{bmatrix} 1 & b+\beta & -be+\gamma \\ 0 & 0 & e+\varepsilon \\ 0 & 0 & 1 \end{bmatrix}$, with arbitrary $b, e, \beta, \gamma, \varepsilon$, $\begin{bmatrix} 0 & \beta & \gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}$ being the (arbitrary) nilpotent.

It is easy to check that for any $M \in \mathcal{T}$, $A_i + M(A_i - A_i^2)$ is an idempotent (of the same type as E_i), and the proof is complete. $\square$

In 7 variables, the subrings $\begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{bmatrix}$, $\begin{bmatrix} \mathbb{Z} & 3\mathbb{Z} & \mathbb{Z} \\ 3\mathbb{Z} & \mathbb{Z} & 3\mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{bmatrix}$ were checked, without success.

Since we do not know of subrings corresponding to *8 variables*, the next step is indeed trying to find a nil-clean 3×3 matrix which is not exchange in the full matrix ring $\mathcal{M}_3(\mathbb{Z})$, the aim of this exposition.

3 Nil-clean up to similarity

Since in the sequel we present results which support the conjecture stated in the introduction, we first describe what should be done in order to prove it.

In order to diminish the huge amount of nil-clean matrices to start with, the following results will be useful.

Lemma 3.1 *The exchange property for elements in any ring is invariant to conjugations.*

Proof. Let $a \in R$ and assume there exists $m \in R$ such that $e = a + m(a - a^2) = e^2$. For any unit $u \in U(R)$ we show that $u^{-1}au$ is also exchange. Indeed, for $x = u^{-1}mu$, a simple computation shows that $u^{-1}au + x[u^{-1}au - (u^{-1}au)^2] = u^{-1}[a + m(a - a^2)]u$ and so the property follows since idempotency is invariant to conjugations. $\square$

As for similarity (the conjugation in square matrix rings), the best result for our purpose is [9]: *every idempotent matrix over a projective-free ring admits a diagonal reduction*. More precisely

Proposition 3.2 *Let R be a commutative ring. Then the following are equivalent:*

(1) *Every nonzero finitely generated projective R-module is free;*

(2) *For any idempotent $E \in \mathcal{M}_n(R)$, there exists $W \in GL_n(R)$ such that $W^{-1}EW = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ for some r.*

Recall that a commutative ring R is projective-free provided that every finitely generated projective R-module is free. For instance, every commutative local ring and every *principal ideal domain* (e.g. $\mathbb{Z}$) are projective-free.

Therefore, since both nil-clean and exchange are similarity invariants, when showing that any 3×3 nil-clean matrix (over a projective-free ring) is exchange, we can suppose that the idempotent is E_{11} or else $E_{11} + E_{22}$.

Having this simplification for the idempotents we have to control the 3×3 nilpotent matrices.

This can be done using the following characterization (see [2])

Proposition 3.3 *Let R be a (commutative) integral domain and let U be an arbitrary matrix in $\mathcal{M}_2(R)$. There is a nilpotent matrix $N \in \mathcal{M}_3(R)$ which has U as the northwest 2×2 corner, whenever there exist elements $a, b, x, y \in R$ such that $ax + by = \det(U) - \mathrm{Tr}(U)^2$ and $bxu_{12} + ayu_{21} - axu_{22} - byu_{11} = \mathrm{Tr}(U)\det(U)$. Such a matrix exists if (e.g.) u_{12} or u_{21} is a unit.*

Conversely, if N is a 3×3 nilpotent matrix which has U as the northwest 2×2 corner, the previous relations hold for $a = n_{13}, b = n_{23}, x = n_{31}$ and $y = n_{32}$.

Summarizing, starting with *any* 2×2 matrix U, we use Proposition 3.3 in order to list the 3×3 nilpotents T obtained by completion, we add E_{11} or $E_{11} + E_{22}$ and for the matrix A respectively A' obtained this way, we have to find an integral 3×3 exchanger M such that $C = A + M(A - A^2) = C^2$.

In the general case, for a given 3×3 (nil-clean) matrix A, in order to find an exchanger M such that $C = A + M(A - A^2) = C^2$ we have to solve a (huge) 9×9 quadratic system.

4 Positive results

Since solving such a 9×9 quadratic systems is out of the question, all that can be done is to study some special cases.

In any special case, the difficulty, for a given class of nil-clean 3×3 matrices, is to find at least one exchanger. In the proofs below, these exchangers appear as a result of some kind of computational algebra. Using suitable programs, a lot of examples were observed and finally the rule was extracted which gives the right exchanger.

The process of covering special cases is slow and difficult. Large room is left for further research.

We first discuss the special cases mentioned in Proposition 3.3, that is, when an off-diagonal entry of U is a unit. Below we cover the cases $u_{12} \in \{\pm 1\}$.

Theorem 4.1 *The following classes of 3×3 nil-clean matrices are exchange:*

(i) $A = E_{11} + T = \begin{bmatrix} u_{11}+1 & 1 & 0 \\ u_{21} & u_{22} & 1 \\ x & m & -u_{11}-u_{22} \end{bmatrix}$.

(ii) $A' = E_{11} + E_{22} + T = \begin{bmatrix} u_{11}+1 & 1 & 0 \\ u_{21} & u_{22}+1 & 1 \\ x & m & -u_{11}-u_{22} \end{bmatrix}$.

(iii) $A = E_{11} + T = \begin{bmatrix} u_{11}+1 & -1 & 0 \\ u_{21} & u_{22} & 1 \\ x & m & -u_{11}-u_{22} \end{bmatrix}$.

(iv) $A' = E_{11} + E_{22} + T = \begin{bmatrix} u_{11}+1 & -1 & 0 \\ u_{21} & u_{22}+1 & 1 \\ x & m & -u_{11}-u_{22} \end{bmatrix}$.

Proof. We start with a 2×2 matrix $U = \begin{bmatrix} u_{11} & 1 \\ u_{21} & u_{22} \end{bmatrix}$, completed to a 3×3 nilpotent $T = \begin{bmatrix} u_{11} & 1 & 0 \\ u_{21} & u_{22} & 1 \\ x & m & -u_{11}-u_{22} \end{bmatrix}$ (i.e. $a = 0$, $b = 1$) where $m = \det U - \mathrm{Tr}^2 U = -u_{11}^2 - u_{22}^2 - u_{11}u_{22} - u_{21}$, $l = \det U \cdot \mathrm{Tr} U = (u_{11}u_{22} - u_{21})(u_{11} + u_{22})$

and $x = l + u_{11}m = (u_{11}u_{22} - u_{21})(u_{11} + u_{22}) - u_{11}(u_{11}^2 + u_{22}^2 + u_{11}u_{22} + u_{21}) = -u_{11}^3 - 2u_{11}u_{21} - u_{21}u_{22}$.

(i) First $A^2 = \begin{bmatrix} (u_{11}+1)^2 + u_{21} & u_{11} + u_{22} + 1 & 1 \\ u_{21} - u_{11}^3 - u_{11}u_{21} & -u_{11}^2 - u_{11}u_{22} & -u_{11} \\ . & . & . \end{bmatrix}$ where the 3rd row will play no rôle.

Secondly, $A - A^2 = \begin{bmatrix} -u_{11}(u_{11}+1) - u_{21} & -u_{11} - u_{22} & -1 \\ u_{11}^3 + u_{11}u_{21} & u_{11}^2 + u_{11}u_{22} + u_{22} & 1 + u_{11} \\ . & . & . \end{bmatrix}$.

Take $M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -u_{22} & 1 & 0 \end{bmatrix}$ and compute $C = A + M(A - A^2) =$

$$\begin{bmatrix} u_{11}+1 & 1 & 0 \\ -u_{11}(u_{11}+1) & -u_{11} & 0 \\ u_{11}^3 + u_{11}^2u_{22} + u_{11}u_{21} + u_{11}u_{22} + u_{21}u_{22} + x & (u_{11}+u_{22})^2 + u_{22} + m & 1 \end{bmatrix}$$

$= \begin{bmatrix} u_{11}+1 & 1 & 0 \\ -u_{11}(u_{11}+1) & -u_{11} & 0 \\ \alpha & \beta & 1 \end{bmatrix}$.

Then $C^2 = \begin{bmatrix} u_{11}+1 & 1 & 0 \\ -u_{11}(u_{11}+1) & -u_{11} & 0 \\ \gamma & \delta & 1 \end{bmatrix}$ and for $C = C^2$ we check $\alpha = \gamma$ and $\beta = \delta$. Here $\gamma = (u_{11}+1)(\alpha - \beta u_{11}) + \alpha = \alpha$ and $\delta = \alpha - \beta u_{11} + \beta = \beta$, because $\alpha = \beta u_{11}$. Indeed, $\alpha = u_{11}^3 + u_{11}^2 + u_{11}u_{21} + u_{11}u_{22} + u_{21}u_{22} - u_{11}^3 - 2u_{11}u_{21} - u_{21}u_{22} = u_{11}(u_{11}u_{22} - u_{21} + u_{22})$ and $\beta = u_{11}u_{22} + u_{22} - u_{21}$.

(ii) We use the same exchanger $M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -u_{22} & 1 & 0 \end{bmatrix}$.

(iii) The only difference is that we use the exchanger $M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ u_{22} & 1 & 0 \end{bmatrix}$.

(iv) We use the same exchanger $M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ u_{22} & 1 & 0 \end{bmatrix}$. $\square$

Moreover, the case u_{21} is a unit, i.e. $u_{21} \in \{\pm 1\}$ is dealt analogously.

Example. (i) For $u_{11} = u_{22} = -3$, $u_{21} = -1$, that is $A = \begin{bmatrix} -2 & 1 & 0 \\ -1 & -3 & 1 \\ 18 & -26 & 6 \end{bmatrix}$ we get $C = \begin{bmatrix} -2 & 1 & 0 \\ -6 & 3 & 0 \\ -21 & 7 & 1 \end{bmatrix} = C^2$.

(ii) $A = \begin{bmatrix} -2 & 1 & 0 \\ -1 & -2 & 1 \\ 18 & -26 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 1 & 0 \\ -6 & 3 & 0 \\ -20 & 10 & 1 \end{bmatrix} = C^2$.

In the sequel, we use the block representation of the 3×3 nilpotent, $T = \begin{bmatrix} U & \alpha \\ \beta & -t \end{bmatrix}$ with $\alpha = \begin{bmatrix} a \\ b \end{bmatrix}$, $\beta = [x\ y]$ and $t = \mathrm{Tr}(U) = u_{11} + u_{22}$.

Next we deal with the case $\alpha = \mathbf{0}$, that is $a = b = 0$.

We came to study this particular case, trying to simplify the computer programs, this way permitting these to consider only the case $a \neq 0 \neq b$ (which was further equivalently simplified to $a, b > 0$).

Replacement in the completion equations shows that such nilpotent completions are possible iff $\det U = \mathrm{Tr}(U) = 0$, i.e. iff $U^2 = 0_2$ (U is a nilpotent 2×2 matrix), and in this case $\beta = [x\ y]$ may be chosen arbitrary. Hence $u_{22} = -u_{11}$, $u_{12}u_{21} = -u_{11}^2$, that is, we discuss matrices of form $\begin{bmatrix} u_{11} & u_{12} & 0 \\ u_{21} & -u_{11} & 0 \\ x & y & 0 \end{bmatrix}$. Notice that since $u_{11}^2 = -u_{12}u_{21}$, u_{11} possibly divides u_{12} or u_{21}.

Theorem 4.2 *Consider the nil-clean matrices* $A = E_{11} + \begin{bmatrix} u_{11} & u_{12} & 0 \\ u_{21} & -u_{11} & 0 \\ x & y & 0 \end{bmatrix}$. *Such matrices are exchange in the following cases:*

(i) $u_{11} = 0$.

(ii) $u_{11} \neq 0$ *and*

1. u_{11} *divides* u_{12}.

2. u_{11} *divides* u_{21}.

3. $u_{12} = 1$ *(or* $u_{21} = 1$*).*

4. $u_{11} = u(u+1)$, $u_{12} = u^2$.

Proof. (i) The case $u_{11} = 0$ is easily settled. Then $u_{22} = 0$ and at least one of u_{12}, u_{21} is zero. Say $u_{21} = 0$. Hence $A = \begin{bmatrix} 1 & u_{12} & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{bmatrix}$ and it is readily checked that for the exchanger $-E_{33}$ we get $C = \begin{bmatrix} 1 & u_{12} & 0 \\ 0 & 0 & 0 \\ x & xu_{12} & 0 \end{bmatrix}$, an idempotent.

(ii) It is readily checked that for $u_{11} \neq 0$ such matrices are *not medium* nil-clean.

Further, $A^2 = \begin{bmatrix} (1+u_{11})^2 + u_{12}u_{21} & u_{12} & 0 \\ u_{21} & 0 & 0 \\ (1+u_{11})x + u_{21}y & u_{12}x + u_{22}y & 0 \end{bmatrix}$ and $A - A^2 =$

$$= \begin{bmatrix} -u_{11} & 0 & 0 \\ 0 & u_{22} & 0 \\ -u_{11}x - u_{21}y & -u_{12}x + (1-u_{22})y & 0 \end{bmatrix} = \begin{bmatrix} -u_{11} & 0 & 0 \\ 0 & -u_{11} & 0 \\ s & t & 0 \end{bmatrix}$$ if we denote $s = u_{22}x - u_{21}y$, $t = -u_{12}x + (1-u_{22})y$.

Using $u_{22} = -u_{11}$ we get $C = A + M(A - A^2) =$

$$= \begin{bmatrix} 1+(1-m_{11})u_{11}+m_{13}s & u_{12}-m_{12}u_{11}+m_{13}t & 0 \\ u_{21}-m_{21}u_{11}+m_{23}s & -(1-m_{11})u_{11}+m_{23}t & 0 \\ x-m_{31}u_{11}+m_{33}s & y-m_{32}u_{11}+m_{33}t & 0 \end{bmatrix}.$$

Multiple computer tests show that we can always chose $m_{13}=m_{23}=0$, $m_{33}=-1$, and $m_{11}+m_{22}=0$. For these values,

$$C = \begin{bmatrix} 1+(1-m_{11})u_{11} & u_{12}-m_{12}u_{11} & 0 \\ u_{21}-m_{21}u_{11} & -(1-m_{11})u_{11} & 0 \\ x-m_{31}u_{11}-s & y-m_{32}u_{11}-t & 0 \end{bmatrix} =$$

$\begin{bmatrix} 1+(1-m_{11})u_{11} & u_{12}-m_{12}u_{11} & 0 \\ u_{21}-m_{21}u_{11} & -(1-m_{11})u_{11} & 0 \\ x(1+u_{11})+u_{21}y-m_{31}u_{11} & u_{12}x-u_{11}y-m_{32}u_{11} & 0 \end{bmatrix}$, which already has $\mathrm{Tr}(C)=1$.

We have to deal with the idempotency of a matrix C of form $\begin{bmatrix} c & u & 0 \\ d & v & 0 \\ e & w & 0 \end{bmatrix}$ where $c+v=1$. It is easily seen that this amounts to the vanishing of the minors $\begin{vmatrix} c & u \\ d & v \end{vmatrix}=0$, $\begin{vmatrix} d & v \\ e & w \end{vmatrix}=0$, that is (since $u_{11}\neq 0$) to

$$[(1-m_{11})^2+m_{12}m_{21}-1]u_{11} = m_{11}+m_{12}u_{21}+m_{21}u_{12}-1 \qquad (1)$$

which, for every m_{11}, is a quadratic Diophantine equation in m_{12}, m_{21} and

$$(1-m_{11})u_{11}m_{31}+(u_{21}-m_{21})m_{32} =$$

$$= (1-m_{11})[x(1+u_{11})+yu_{21}]-x(u_{11}+m_{21}u_{12})-y(u_{21}-m_{21}u_{11}) \qquad (2)$$

which is a linear Diophantine equation in m_{31}, m_{32} (after replacing the solutions found for (1)).

Using (1) and (2), an exchanger was found in the cases given in our statement.

1. Suppose $u_{12}=ku_{11}$. For $M=\begin{bmatrix} 1 & k & 0 \\ 0 & -1 & 0 \\ 0 & kx-y & -1 \end{bmatrix}$ we get the idempotent $C=\begin{bmatrix} 1 & 0 & 0 \\ u_{21} & 0 & 0 \\ x(1+u_{11})+u_{21}y & 0 & 0 \end{bmatrix}=C^2$ (matrices of type $\begin{bmatrix} 1 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{bmatrix}$ are idempotent for any a,b).

2. Suppose $u_{21}=ku_{11}$. For $M=\begin{bmatrix} 1 & 0 & 0 \\ k & -1 & 0 \\ x+k^2 & -y & -1 \end{bmatrix}$ we get $C=\begin{bmatrix} 1 & u_{12} & 0 \\ 0 & 0 & 0 \\ x & u_{12}x & 0 \end{bmatrix}$ $=C^2$ (matrices of type $\begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 0 \\ b & c & 0 \end{bmatrix}$ are idempotent iff $ab=c$).

3. If $u_{12}=1$ then $u_{21}=-u_{11}^2$. This case is already covered by Proposition 4.1. An exchanger is $M=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ u_{11} & 1 & 0 \end{bmatrix}$.

4. In this case $u_{21} = -(u+1)^2$, i.e., $A = \begin{bmatrix} u(u+1)+1 & u^2 & 0 \\ -(u+1)^2 & -u(u+1) & 0 \\ x & y & 0 \end{bmatrix}$ with arbitrary x, y and an exchanger is $M = \begin{bmatrix} 1-u & -u^3 & 0 \\ -1 & u-1 & 0 \\ x & -[1-u(u+1)]y & -1 \end{bmatrix}$ (see remark 2 below, for more details). □

Remarks. 1) Since $u_{12}u_{21} = -u_{11}^2$, the remaining case when u_{11} does not divide u_{12} nor u_{21}, should be $u_{12}^2 = \pm q^2 p$, $u_{21} = \mp pr^2$ and $u_{11} = qpr$ with p, q, r two by two coprime (here $\gcd(u_{12}; u_{21}) = p$ and we may assume $q \neq \pm 1 \neq r$).

So far, it is left open.

2) The special case **4** above arises from a more general situation. Suppose $\gcd(u; v) = 1$. Then also $\gcd(u^2; v) = 1$ and so there exist integers s, t such that $su^2 + tv = 1$. *Very often* we can build an exchanger by choosing $m_{11} = 2 - v$ and $m_{21} = -s$.

Actually, here (1) is

$$(m_{11}^2 - 2m_{11} + m_{12}m_{21})uv - u^2 m_{21} + v^2 m_{12} + 1 - m_{11} = 0$$

and (2) becomes

$$(1 - m_{11})um_{31} - (v + m_{21}u)m_{32} =$$

$$= (1 - m_{11})(xu - yv) - ux + (v + m_{21}u)y + x\frac{1 - m_{11} - m_{21}u^2}{v}$$

where v must divide $1 - m_{11} - m_{21}u^2$. This explains why we chose $m_{21} = -s$: now $1 - m_{11} - m_{21}u^2 = (1-t)v$.

With the selection $m_{11} = 2 - v$ and $m_{21} = -s$, (1) becomes just

$$m_{12}(-su + v) = -uv(v-2) - 1 + t.$$

Since there is no uniqueness for the pair (s, t) it remains to chose a suitable such pair in order to obtain an integer for m_{12}. It is well-known, that if (s, t) is a solution for $su^2 + tv = 1$, all the other solutions are given by $(s + kv, t - ku^2)$. Again, *very often* we can chose (s_0, t_0) with s_0 the least positive s, solution for $su^2 + tv = 1$.

A special case is precisely **4** in the previous theorem.

This also works, for instance, for $(u, v) \in \{(2, 7), (5, 3)\}$ but does not for $(u, v) = (4, 7)$. The later has $A = \begin{bmatrix} 29 & 16 & 0 \\ -49 & -28 & 0 \\ x & y & 0 \end{bmatrix}$ and $s \cdot 16 + t \cdot 7 = 1$ gives $s_0 = 4$ ($t_0 = -9$), $m_{11} = -5$, $m_{21} = -4$. Then $m_{12}(-16 + 7) = -141 - 9 = -150$, has no (integer) solution.

3) However, the last matrix is covered by another class of exchangers: those with $(2u-v)^2 = 1$. Indeed, in this case, an exchanger is of form $M = \begin{bmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ . & . & -1 \end{bmatrix}$. This covers $(u,v) \in \{(2,3); (3,5); (3,7); (4,7); (5,9), ...\}$, but not $(2,7)$.

5 A different attempt

Here we used a different strategy: starting with a 3×3 matrix A, known as *not being exchange*, a program verifies whether a 3×3 matrix T exists, such that $T^2 \neq 0_3 = T^3$ and $A - T$ is idempotent (i.e. A is nil-clean).

In this procedure, since a characterization of not exchange 3×3 matrices is not known so far, we used the following generations of such matrices.

In [6] we can find the following

Theorem 5.12 *Let e be an idempotent in a ring R and $a = b + \varepsilon$ with $b \in S := eRe$, $\varepsilon \in \mathrm{Idem}(\overline{e}R\overline{e})$. Then a is exchange in R iff b is exchange in eRe.*

There are two special cases which are related with our research.

1) $R = \mathcal{M}_3(\mathbb{Z})$ with $e = \mathrm{diag}(1,1,0) = E_{11}+E_{22}$ and then $\overline{e} = \mathrm{diag}(0,0,1) = E_{33}$. In this case we identify $S = eRe$ with $\mathcal{M}_2(\mathbb{Z})$ and $\overline{e}R\overline{e}$ with $\mathbb{Z}$.

2) $R = \mathcal{M}_3(\mathbb{Z})$ with $e = \mathrm{diag}(1,0,0) = E_{11}$ and then $\overline{e} = \mathrm{diag}(0,1,1) = E_{22} + E_{33}$. In this case we identify $S = eRe$ with $\mathbb{Z}$ and $\overline{e}R\overline{e}$ with $\mathcal{M}_2(\mathbb{Z})$.

This way we obtain the following two consequences.

Corollary 5.1 *Let $U \in \mathcal{M}_2(\mathbb{Z})$ and $\varepsilon \in \{0,1\} \subset \mathbb{Z}$. Then $A = \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & \varepsilon \end{bmatrix}$ is exchange in $\mathcal{M}_3(\mathbb{Z})$ iff U is exchange in $\mathcal{M}_2(\mathbb{Z})$.*

Corollary 5.2 *Let $b \in \mathbb{Z}$ and $E = E^2 \in \mathcal{M}_2(\mathbb{Z})$. Then $A = \begin{bmatrix} b & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix}$ is exchange in $\mathcal{M}_3(\mathbb{Z})$ iff b is exchange in $\mathbb{Z}$ iff $b \in \{-1,0,1,2\} \subset \mathbb{Z}$.*

When searching for a nil-clean 3×3 integral matrix which is not exchange, Corollary 5.2 *is not useful*. Indeed, if E is idempotent in $\mathcal{M}_2(\mathbb{Z})$, then $\mathrm{Tr}(A) \geq 3$ or $\mathrm{Tr}(A) \leq 0$, so A is not nil-clean.

As for Corollary 5.1, we should know the 2×2 matrices which are *not exchange*, but since we are looking only for nil-clean 3×3 matrices, we need: if $\varepsilon = 0$, only 2×2 matrices which are *not exchange* with trace 1 or 2, if $\varepsilon = 1$, only 2×2 matrices which are *not exchange* with trace 0 or 1.

From [4] we mention the following

Corollary 5.3 *The following 3×3 matrices are not exchange for any $n \in \mathbb{Z}$, $n \geq 2$:*

(a) $\begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & \varepsilon \end{bmatrix}$ *for* $U \in \begin{bmatrix} n\mathbb{Z}+1 & n\mathbb{Z} \\ n\mathbb{Z} & n\mathbb{Z}+1 \end{bmatrix}$, $\det(U) \notin \{\pm 1\}$ *and* $\varepsilon \in \{0,1\}$,

(b) $\begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & \varepsilon \end{bmatrix}$ *for* $U \in \mathcal{M}_2(n\mathbb{Z})$, $\det(U - I_2) \notin \{\pm 1\}$ *and* $\varepsilon \in \{0,1\}$,

(c) $\begin{bmatrix} b & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix}$ *with any 2×2 idempotent E and* $b \in \mathbb{Z} - \{-1, 0, 1, 2\}$.

Thousands of matrices generated as in this corollary were checked. None was nil-clean.

A program which verifies this is described in the following section.

Many *other attempts* were made in order to find a nil-clean 3×3 matrix which is not exchange.

We just outline here, two of these.

1. Replacing in [2] only the idempotent with $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we get $A = \begin{bmatrix} 2 & \mathbf{0} & 0 \\ -1 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix}$ and $A - A^2 = \begin{bmatrix} -2 & 0 & 0 \\ 2 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix} = A(I_3 - A) = 2C$ which suggests the following plan for searching an example of nil-clean 3×3 matrix which is not exchange: *to find a nil-clean matrix A, which has trace $= 1$ and an odd (diagonal) minor of order two (so rank $\neq 1$) such that $A - A^2 = 2C$ for some integral matrix C.*

Indeed, it is easy to show that if $A \in \mathcal{M}_3(\mathbb{Z})$ with $\mathrm{Tr}(A) = 1$, $A - A^2 = 2C$ for some integral matrix C and A has an odd (diagonal) minor of order two then A is not exchange.

However, a long proof shows that there is no nil-clean 3×3 matrix with the properties above.

2. We can nilpotent complete $U = \begin{bmatrix} 1 & 3 \\ -2 & -4 \end{bmatrix}$ with $\alpha = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and it gives the nilpotents $\begin{bmatrix} 1 & 3 & 2 \\ -2 & -4 & -1 \\ -15 & -23 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & -2 \\ -2 & -4 & 1 \\ 15 & 23 & 3 \end{bmatrix}$. Among the (few) exchangers found, we mention $M = \begin{bmatrix} \mathbf{-3} & \mathbf{4} & -5 \\ \mathbf{1} & \mathbf{-2} & 2 \\ 2 & -1 & -2 \end{bmatrix}$ for the first and $M = \begin{bmatrix} \mathbf{-3} & \mathbf{4} & 5 \\ \mathbf{1} & \mathbf{-2} & -2 \\ -2 & 1 & -2 \end{bmatrix}$ for the second.

Notice that the upper-left 2×2 block is obtained by a rotation of U if we neglect some signs and α and β are interchanged.

A program was designed in order to check whether this does happen in general (and this way, would yield a general proof). Unfortunately, this does not happen in general.

6 Experimentation software

The previous section describes the procedures we need.

Since $\mathbb{Z}$ is an infinite ring, the program cannot exhaust all the 2×2 (or 3×3) integral matrices, to start with. All we can do is start with a limitation for the entries of the matrices. The limit was denoted by (a positive integer) z, and this means that $-z \leq u_{ij} \leq z$ for $U = (u_{ij})$.

First about the procedure (called **A**) which forms all 3×3 nil-clean matrices and checks these for the exchange property.

One version follows Proposition 3.3. In order to have a (theoretically) exhaustive procedure, we have to list all 3×3 nilpotent matrices, obtained by completion. That is why we have to consider the compatibility over $\mathbb{Z}$ of the system

$$\begin{aligned} ax + by &= m \\ (bu_{12} - au_{22})x + (au_{21} - bu_{11})y &= l \end{aligned}$$

where $m = \det(U) - \mathrm{Tr}^2(U)$ and $l = \det(U) \cdot \mathrm{Tr}(U)$.

For a given z, the program writes all 2×2 (integral) matrices U and all (integral) columns $\begin{bmatrix} a \\ b \end{bmatrix}$. Then it finds solutions (x, y) for the above system, that is, constructs all the nilpotent 3×3 completions for U, a, b. Then it adds E_{11} respectively $E_{11} + E_{22}$ and checks the resulting matrices A, A' for the exchange property.

Alternatively, and this is the version which finally was chosen, for a given z, the program writes all 3×3 (integral) matrices, selects the matrices T such that $T^3 = 0_3 \neq T^2$. Then it adds E_{11} respectively $E_{11} + E_{22}$ and checks the resulting matrices A, A' for the exchange property.

Remark. For any given pair $(a, b) \in \mathbb{Z}^2$, the *completion equations* above, form a system of two linear Diophantine equations with unknowns x, y. This system can be discussed with the Cramer rule and with the compatibility of a linear Diophantine equation, but the resulting program is too complicated.

Secondly, about the procedure (called **B**) which starts with not exchange 3×3 matrices and checks the for the nil-clean property. This was based on Corollary 5.3, stated in the previous section. Corresponding to (a) or (b), for a given n and a given z, these not exchange 3×3 matrices are checked for the nil-clean property. As for (c), first all (modulo z) idempotent 2×2 are constructed, then these are completed to 3×3 with a not exchange integer b. Subtracting the nilpotent 3×3 (integral) matrices, the differences are checked for idempotency.

The computer programs performed a series of tests. The tests were done with the purpose to find counter-examples or suggest reasoning pathways for the conjecture.

The search for matrices M is performed incrementally with the nonnegative integer z starting at 0 and incremented by 1 for as long as it is deemed necessary. For each distinct value of z, only the matrices M with all entries in the closed interval $[-z, z]$ and with at least one entry of absolute value equal to z are considered. This **iterative search procedure** has the advantage of splitting the set of all integral matrices M into distinct subsets, covered one at a time.

1. For a given matrix A of order n, with entries in the ring $\mathbb{Z}_k$ (integers modulo k), print all the exchangers M, that is, $C = A + M(A - A^2) = C^2$. All the computations were done in $\mathbb{Z}_k$. The matrices M are searched exhaustively.
2. For a given matrix A of order 3, print all the exchangers M.
3. For a given matrix A of order 3, print the total number of exchangers M and each such matrix M that has at least five zero entries. The search is limited to all matrices C with entries in absolute value less than a given threshold z.
4. Find all exchangers M of order 3, with A subject to various constraints. The iterative search procedure for the matrix M described in *item 2* is performed here.
5. For a given matrix U of order 2, construct the matrices A and A' of order 3 by nilpotent completion in order to get a nilpotent matrix T with $T^2 \neq 0$ and $T^3 = 0$. Then find all exchangers M such that $A^2 \neq A$. The iterative search procedure for the matrix M described in *item 2* is performed here.
6. As a variation, for all the matrices U of order 2 with entries in absolute value limited by a given threshold, and with $u_{12} = \pm 1$ use the procedure described in *item 5*.
7. As another variation, print all the matrices U of order 2, and the corresponding matrices A and A' for which there is no exchanger M with entries in absolute value limited by a given threshold, such that $A^2 \neq 0_3$.
8. For a given matrix A of order 3, print all matrices T such that $T^3 = 0$ and $A - T = C = C^2$. The iterative search procedure for the matrix T described in *item 2* is performed here.

Our final program followed the general procedure described in Section 5. Basically, we needed to verify the following:

9. For all matrices U of order 2, with entries limited by a given integer k, and for all integers $a > 0$ and $b > 0$ also limited by k, verify that for each of the matrices A and A' of order 3 completed as outlined in Section 5 there exists an exchanger M.

z	input	exchange	output	input%	exchange%	total%
init	43218	38054	5164	100%		88.05%
0	5164	0	5164	11.95%	0%	0%
1	5164	2189	2975	11.95%	42.39%	42.39%
2	2975	1347	1628	6.88%	45.28%	68.47%
3	1628	703	925	3.77%	43.18%	82.09%
4	925	352	573	2.14%	38.05%	88.90%
5	573	167	406	1.33%	29.14%	92.14%
6	406	134	272	0.94%	33.00%	94.73%

Table 1 Results for the program at *item 9*.

Because of the workload involved by such a testing, we decided to work incrementally, and consider the list of all combinations $(U, a, b, A/A')$ for which an M was not yet determined. As such, at every iteration of the procedure, the program produces the list of all combinations $(U, a, b, A/A')$ that do not have an M determined so far. The program actually verified the following:

10. Given a list of all combinations $(U, a, b, A/A')$ to be evaluated, and given an integer z, print all combinations from the list above for which there is no exchanger M with the maximal absolute value of entries equal to z.

Out of all the combinations $(U, a, b, A/A')$ to be evaluated, the program only deals with those verifying some validity hypotheses, as mentioned in Section 5.

For matrices U limited by $k = 3$, this program, running incrementally, leads to the results in Table 1. There, for each value of z, the column **input** indicates the total number of matrices tested for exchange, the column **exchange** indicates the total number of matrices found to be exchange and the column **output** indicates the total number of matrices not yet found to be exchange. Further, column **input%** indicates the percentage of all the matrices that are tested at this phase, column **exchange%** indicates the percentage of matrices found to be exchange out of all the matrices tested at this phase, and column **total%** indicates the percentage of matrices found to be exchange so far out of all the total number of matrices verifying the validity hypotheses.

The line marked **init** shows the prefilter phase, and the data should be red differently: out of 43128 matrices, 38054 did not verify the validity hypotheses, representing 88.05% of the total.

The Table 2 shows the number of matrices for various values of k that verify the validity hypotheses.

11. For all matrices T of order 3, with entries limited by a given integer k, and $T^3 = 0 \neq T^2$, verify that for each of the matrices $A = T + E_{11}$ and $A' = T + E_{11} + E_{22}$ there exists an exchanger M.

The same incremental approach has been used here. Our results for values of k equal to 3, 4, 5, 6 are given in Tables 3, 4, 5, and 6 respectively.

k	total	valid	[%]
1	162	20	12.35%
2	5,000	784	15.68%
3	43,218	5,164	11.95%
4	209,952	19,408	9.24%
5	732,050	48,332	6.60%
6	2,056,392	111,800	5.44%
7	4,961,250	201,420	4.06%
8	10,690,688	376,144	3.52%
9	21,112,002	597,092	2.83%
10	38,896,200	941,200	2.42%

Table 2 Number of matrices verifying the validity hypotheses of *item 9*.

total matrices T	40,353,607	
valid matrices T	12,096	0.02998%
total matrices A, A'	24,192	
matrices not exchange, $z = 1$	1,312	5.42328%
matrices not exchange, $z = 2$	128	0.52910%
matrices not exchange, $z = 3$	0	0.00000%

Table 3 Results for the program at *item 10* for $k = 3$

total matrices T	387,420,489	
valid matrices T	38,160	0.00985%
total matrices A, A'	76320	
matrices not exchange, $z = 1$	9,984	13.08176%
matrices not exchange, $z = 2$	1,824	2.38994%
matrices not exchange, $z = 3$	336	0.44025%
matrices not exchange, $z = 4$	16	0.02096%
matrices not exchange, $z = 5$	0	0.00000%

Table 4 Results for the program at *item 10* for $k = 4$

total matrices T	2,357,947,691	
valid matrices T	73,536	0.00312%
total matrices A, A'	147072	
matrices not exchange, $z = 1$	25,616	17.41732%
matrices not exchange, $z = 2$	5,728	3.89469%
matrices not exchange, $z = 3$	1,312	0.89208%
matrices not exchange, $z = 4$	192	0.13055%
matrices not exchange, $z = 5$	16	0.01088%
matrices not exchange, $z = 6$	0	0.00000%

Table 5 Results for the program at *item 10* for $k = 5$

12. For all matrices T of order 3, with entries limited by a given integer k, and for a certain given value of n, satifying the hypotheses of Corrolary 10, verify that there is no matrix T such that $T^3 = 0$ and $A - T = C = C^2$, with values limited by a given integer z. The iterative search procedure for the matrix T described in *item 2* is performed here.

total matrices T	10,604,499,373	
valid matrices T	137,472	0.00130%
total matrices A, A'	274,944	
matrices not exchange, $z = 1$	70,480	25.63431%
matrices not exchange, $z = 2$	23,328	8.48464%
matrices not exchange, $z = 3$	6,848	2.49069%
matrices not exchange, $z = 4$	3,104	1.12896%
matrices not exchange, $z = 5$	976	0.35498%
matrices not exchange, $z = 6$	...	...
matrices not exchange, $z = 7$	...	...

Table 6 Results for the program at *item 10* for $k = 6$

Acknowledgements All the experiments in this paper were performed with C++ programs written by the second author, and running on a computer server located with the Department of Computer Science, Babeş-Bolyai University in Cluj, Romania.

References

1. D. Andrica, G. Călugăreanu *A nil-clean 2 x 2 matrix over integers which is not clean*. J. of Algebra and its Appl., vol. **13**, (6) (2014), 9 pages.
2. G. Călugăreanu *Some matrix completions over integral domains*. Linear Algebra and its Appl. (2016).
3. G. Călugăreanu *A new class of nil-clean elements which are exchange*. Beiträge zur Algebra und Geometrie (2019).
4. G. Călugăreanu, H. F. Pop *Trace 1,* 2×2 *matrices over principal ideal domains are exchange*. submitted.
5. A. J. Diesl *Classes of strongly clean rings*. PhD Thesis, University of California, Berkeley (2006).
6. D. Khurana, T. Y. Lam, P. Nielsen *Exchange elements in rings, and the equation* $XA - BX = I$. Trans. Amer. Math. Soc. **369** (2017), 495-516.
7. W. K. Nicholson *Lifting idempotents and exchange rings*. Trans. Amer. Math. Soc., **229** (1977), 269-278.
8. W. K. Nicholson *On exchange rings*. Comm. in Algebra **25** (6) (1997), 1917-1918.
9. A. Steger *Diagonalibility of Idempotent Matrices*. Pacific J. of Math. **19** (3) (1966), 535-542.

Received: / Accepted:

Authors

Grigore Călugăreanu (Corresponding author),
Department of Mathematics, Faculty of Mathematics and Computer Science,
Babeş-Bolyai University,
Cluj-Napoca, Romania
E-mail: calu@math.ubbcluj.ro

Horia F. Pop,
Department of Computer Science, Faculty of Mathematics and Computer Science,
Babeş-Bolyai University
Cluj-Napoca, Romania
E-mail: hfpop@cs.ubbcluj.ro