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The structure of indecomposable sigma-pure-injective modules

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A module N is Σ-*pure-injective* iff $N^{(\aleph_0)}$ is pure-injective (equationally compact). Equivalently, N is Σ-pure-injective iff its complete first-order theory is *totally transcendental* ("tt" for short, and hereafter). Every tt module can be written in an essentially unique way as a direct sum of indecomposable tt modules, so the only important structural questions are those about the indecomposables. A good example of a structure theorem for a class of tt modules is the classic theorem of E. Matlis [1958] describing the indecomposable injective modules over a commutative noetherian ring. One might ask how far such a structure theory might be extended to arbitrary indecomposable tt modules. Examples even from injectives over non-commutative noetherian rings show how difficult (and even poorly posed) this question is.

One possible tool in such a study was introduced by Ivo Herzog in 1993: the *elementary socle series* $(\mathrm{soc}^{\alpha}(N))_{\alpha\in\mathrm{Ord}}$. This is a first-order definable analogue of the socle series. Although defined only as an increasing sequence of subgroups, in fact each term of the series is a fully invariant submodule of N, and for some ordinal α_0, $N = \mathrm{soc}^{\alpha_0}(N)$. In the case of an indecomposable injective over a commutative noetherian ring, this is exactly the hierarchy of Matlis, but even for an injective over a well-behaved non-commutative noetherian ring, it need not correspond to the "fundamental series" of Jategaonkar. Very little more is known in general about the properties of the elementary socle series.

There are several reasonable questions to ask:

1. What kind of bi-module is $\mathrm{soc}^1(N)$?
2. What are the properties of the series $\left(\mathrm{soc}^{\alpha+1}(N)/\mathrm{soc}^{\alpha}(N)\right)_{\alpha<\alpha_0}$?
3. What are the properties of the series $(N/\mathrm{soc}^{\alpha}(N))_{\alpha<\alpha_0}$?

The purpose of this talk is not only to explain what might be the beginnings of an interesting structure theory, but to seek input on new directions to explore. Those examples which are understood well enough to give answers to the above questions depend on special features which obscure the general nature of the questions. For instance, I do not know of any example where $N/\mathrm{soc}^1(N)$ is not tt; but this is almost certainly an artifact of the examples that I have been studying.