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Algebra schemes and their representations

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The equivalence (*Cartier duality*) between the category of topologically flat formal k-groups and the category of flat bialgebras has been treated as a duality of continuous vector spaces (of functions). This is owing to the fact that the reflexivity of vector spaces of infinite dimension does not hold if one does not provide them with a certain topology and does not consider the continuous dual. However, we obtain this duality without providing the vector spaces of functions with a topology. Let R be a commutative ring with unit. It is natural to consider R-modules as R-module functors in the following way: if E is an R-module, let $\mathbf{E}$ be the R-module functor defined by $\mathbf{E}(B) := E \otimes_R B$ for every R-algebra B which belongs to the category $\mathcal{C}_R$ of R-algebras. Now, if F is a functor of R-modules, its dual F^* can be defined in a natural way as the functor of R-modules defined $F^*(B) := \mathrm{Hom}_B(F_{|B}, \mathbf{B})$. In this work we will prove that the functor defined by an R-module is reflexive: $\mathbf{E} \xrightarrow{\sim} \mathbf{E}^{**}$, even in the case of R being a ring.

We call the functors $\mathbf{E}^*$ R-module schemes and if they are R-algebra functors too, we will say they are R-algebra schemes. P. Gabriel proved that the category of topologically flat formal R-varieties is equivalent to to the category of flat cocommutative R-coalgebras, where R is a pseudocompact ring. We prove that the category of R-algebra schemes is equivalent to the category of R-coalgebras, where R is a ring.

From this perspective, on the theory of algebraic groups and their representations R-module schemes appear in a necessary way, as also do R-algebra schemes as linear envelopes of groups. Let $G = \mathrm{Spec}\, A$ be an R-group and let $G^{\cdot}$ be the functor of points of G, i.e., $G^{\cdot}(B) = \mathrm{Hom}_{R-\text{schemes}}(\mathrm{Spec}\, B, G)$ for all $B \in \mathcal{C}_R$, and let $R[G^{\cdot}]$ be the "linear envelope of $G^{\cdot}$". We prove that the R-algebra scheme closure of $R[G^{\cdot}]$ is the R-algebra scheme $\mathbf{A}^*$, and the category of G-modules is equal to the category of $\mathbf{A}^*$-modules. Therefore, the theory of linear representations of a group $G = \mathrm{Spec}\, A$ is a particular case of the theory of $\mathbf{A}^*$-modules.

Finally, we prove that every R-algebra scheme $\mathbf{A}^*$ is an inverse limit of finite R-algebra schemes. We characterize the separable algebra schemes and we prove the theorem of Wedderburn-Malcev in the context of algebra schemes. We apply this theory of algebra schemes to the theory of algebraic groups, with special emphasis on the Cartier duality and the invariant theory.